SET THEORY

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In everyday life, we often talk of the collection of objects such as a bunch of keys, flock of birds, pack of cards, etc. In mathematics, we come across collections like natural numbers, whole numbers, prime and composite numbers. The modern study of set theory was initiated by Georg Cantor and Richard Dedekind in the 1870s.

A set is a well-defined collection of distinct objects. The objects that make up a set are known as the elements or members of that set. Members can be anything: numbers, people, letters of the alphabet etc. *Sets are conventionally denoted with capital letters and its members are enclosed by curly brackets.* e.g.

$\mathbb{N}$ = Set of natural numbers = $\{1, 2, 3, \dots, \infty\}$

$\mathbb{W}$ = Set of whole numbers = $\{0, 1, 2, 3, \dots, \infty\}$

$\mathbb{Z}$ = Set of integers = $\{-\infty, \dots, -3, -2, -1, 0, 1, 2, 3, \dots, \infty\}$

$\mathbb{Z}^+$ = Set of +ve integers = $\{1, 2, 3, \dots, \infty\}$

$\mathbb{R}$ = Set of real numbers

$\mathbb{Q}$ = Set of rational numbers

$\mathbb{C}$ = Set of complex numbers

The term 'well-defined' is very important in the definition of set. E.g. we want to prepare the collection of bad boys in a particular class. But here is no specific rules by which we can tell that a student is good or bad. So this collection is not well-defined. So this collection is varied from teacher to teacher, i.e. we say that this collection is not well-defined. So this collection does not make any set.

Members of set:

$B = \{a, b, c, \dots, x, y, z\}$. Here x is one of the objects/members of B. This is denoted as $x \in B$ and is read as "x belongs to B". Also 1 is not a member of above set B, so we say $1 \notin B$ which is read as "1 not belongs to B".

We can represent a set in three ways. Here we illustrate it with examples.

i) ***Statement form**:* E.g. Let A be a set of natural numbers between 1 and 5.

ii) ***Roster or tabular form**:* In roster form, all the elements are listed, the elements are being separated by commas and are enclosed with curly braces. The order in which elements are listed is immaterial but elements must not be repeated. The dots at the end tells us that set continues infinitely. E.g. A={1,2,3,4,5}

iii) ***Set builder form**:* Here all the elements of a set, must possess a single common property which is not possessed by any element outside it. In this form, the elements of the set is described by a single variable 'x' or any other variable followed by '**:**' or '**|**' (this means: such that) and then write the single common property possessed by all the elements of that set and enclose the whole description in braces. E.g. $A = \{x : x \in \mathbb{N}\ and\ 1 \leq x \leq 5\}$

Null Set / Void Set / Empty Set:

A set having no member is called null set & is denoted by $\{\}$ or $\emptyset$. E.g. $A = \{x : x \in \mathbb{N}\ and\ 5 < x < 6\}$ or $A = \{x : x \in \mathbb{N}\ and\ 2x - 1 = 0\}$

Singleton Set:

A set which contains only one element is called a singleton set. E.g. $A = \{x : x \in \mathbb{N}\ and\ x^2 = 4\}$

Finite & infinite set:

A set which is empty or contains a finite number of elements is called finite set, otherwise the set is called infinite set. All infinite sets cannot be expressed in roster form. E.g. $\mathbb{R}$ can't be expressed in roster form however $\mathbb{N}$ can.

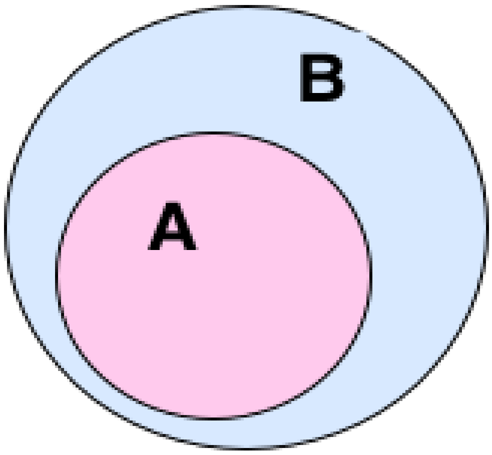
Cardinal Number of a Set:

The number of distinct elements in a given set A is called the cardinal number of set A and is denoted by $n(A)$. E.g. if $= \{x : x \in \mathbb{N}\ and\ (x-1)(x-2) = 0\}$, then $n(A) = 2$

Equivalent sets:

Two sets A and B are said to be equivalent if their cardinal number is same, i.e., $n(A) = n(B)$. The symbol for denoting an equivalent set is '$\leftrightarrow$'. E.g. $A = \{1, 2, 3\}, B = \{p, q, r\}$. Here $n(A) = n(B) = 3$ .Therefore, A $\leftrightarrow$ B

Subset & Superset:



A set A is said to be a subset of set B if every member of set A is also a member of set B. This is written as A ⊆ B.

Mathematically, if for all $x \in A \Longrightarrow x \in B$ then we say $A \subseteq B$ and vice versa.

And B is called superset of A & is written as $B \supseteq A$.

E.g. The sets N, Z^+, Z^- are all the subsets of R.

The empty set is a subset of every set and every set is a subset of itself. i.e. $\emptyset \subseteq B, B \subseteq B$

If $B = \{1,2,3,4\}$ then the subsets of B are ∅, {1}, {2}, {3}, {4}, {1,2}, {1,3}, {1,4}, {2,3}, {2,4}, {3,4}, {1,2,3}, {1,2,4}, {1,3,4}, {2,3,4}, {1,2,3,4}. If B has n members, then number of subsets of B are 2^n.

Proper subsets:

If A is a subset of B, but not equal to B, then A is called a proper subset of B and is written as $A \subset B$. If a set B has n members, then number of proper subsets of B is $2^n - 1$.

Disjoint sets:

Two sets A & B are said to be disjoint if they have no common members. If A & B are two disjoint sets, then $A \cap B = \emptyset$

Power set:

The set of all subsets of B is called power set of B and is written as $P(B)$. If $B = \{1,2,3\}$ then $P(B) = \{\{1, 2, 3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1\}, \{2\}, \{3\}, \emptyset\}$

Equality of two sets:

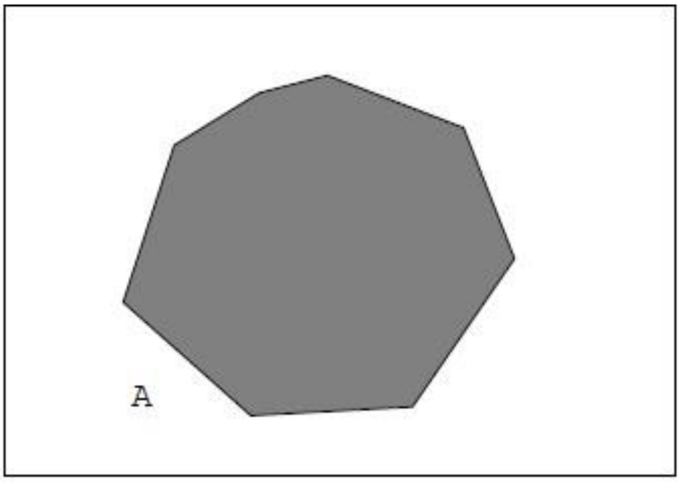
Two sets A and B are said to be equal if they contain the same elements. *Every element of set A is an element of set B and every element of set B is also an element of set A.* E.g. if $A = \{p, q, r, s\}$ and $B = \{p, s, r, q\}$, then $A = B$. Mathematically, two sets A and B are equal if $A \subseteq B$ and $B \subseteq A$.

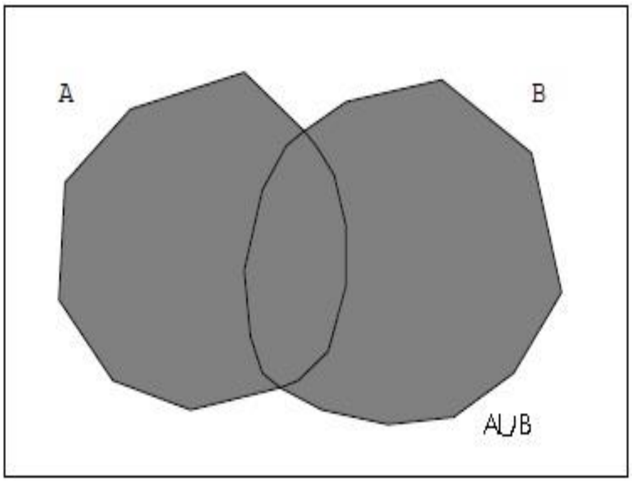
Universal set:

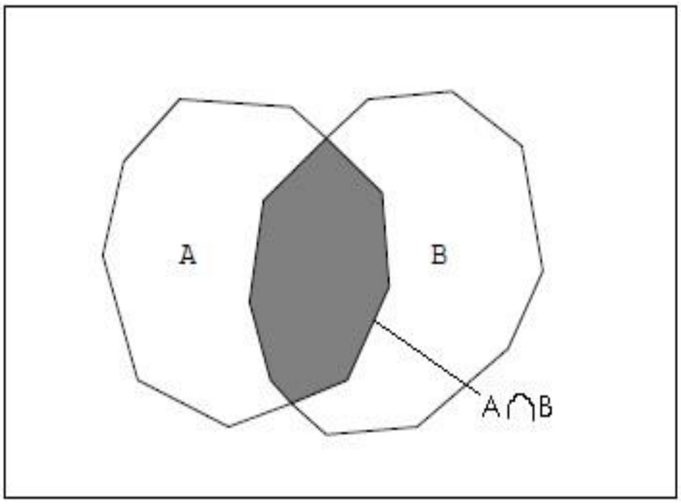
A universal set is the collection of all objects in a particular context or theory. All other sets in that framework are the subsets of the universal set. It is denoted by U.

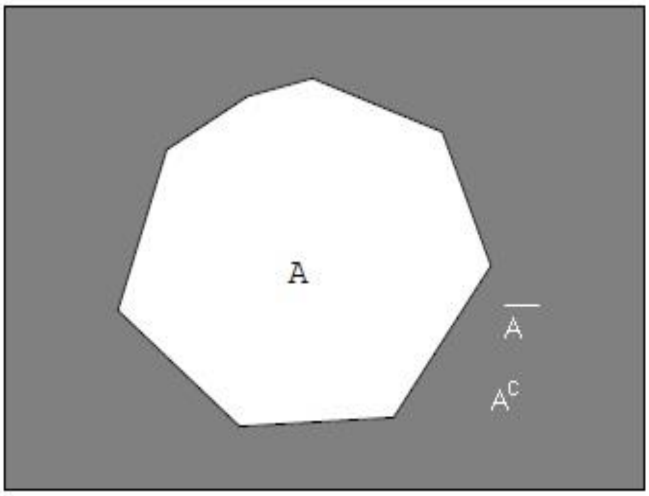
Venn Diagrams:

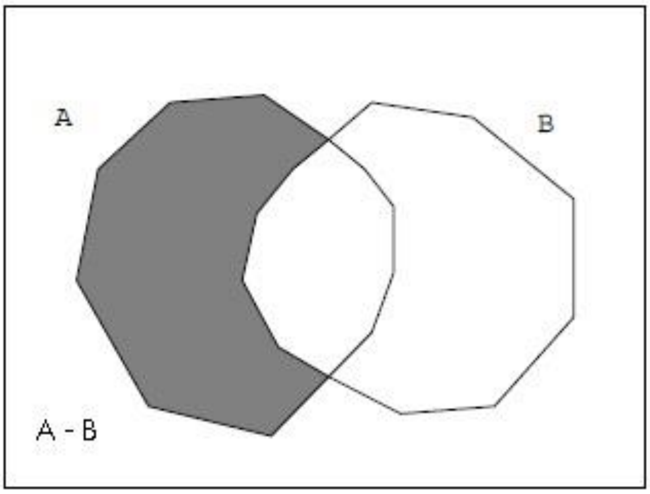
Venn diagrams are graphic representations of sets as enclosed areas in the plane.

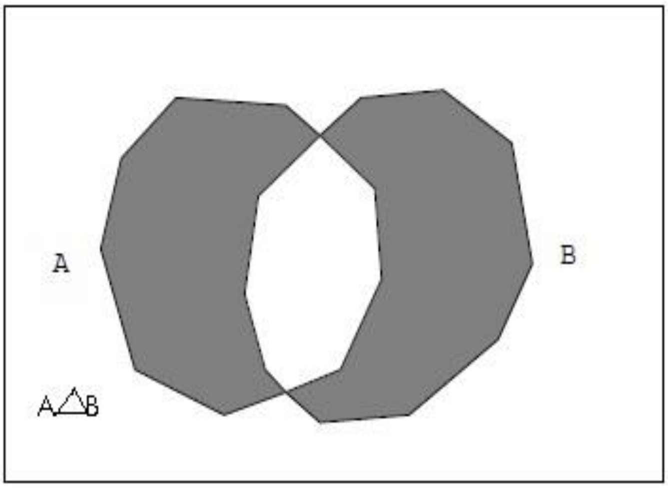












Set operations:

1) ***Union:*** The set of elements that belong to either of two sets. $A \cup B = \{x \mid x \in A \ or \ x \in B \}$

2) ***Intersection:*** The common elements of two sets. $A \cap B = \{x \mid x \in A \ and \ x \in B \}$. If $\cap B = \emptyset$, then the sets are said to be disjoint. In set algebra, $A \cap B$ is same as AB.

3) ***Complement:*** The set of elements (in the universal set) that do not belong to a given set. $A^c = \bar{A} = \{x \mid x \in U \ and \ x \notin A\} = U - A$

4) ***Difference of two sets:*** The set of elements that belong to a set but not to another. $A - B = \{x \mid x \in A \ and \ x \notin B\} = \{x \mid x \in A \ and \ x \in B^c\} = A \cap B^c$

5) ***Symmetric difference:*** Symmetric difference of two sets is the set of elements which are in either of the sets and not in their intersection. The symmetric difference of the sets *A* and *B* is commonly denoted by $A \triangle B$ or $A \ominus B$.or $A \oplus B$. $A\Delta B = (A - B) \cup (B - A)$

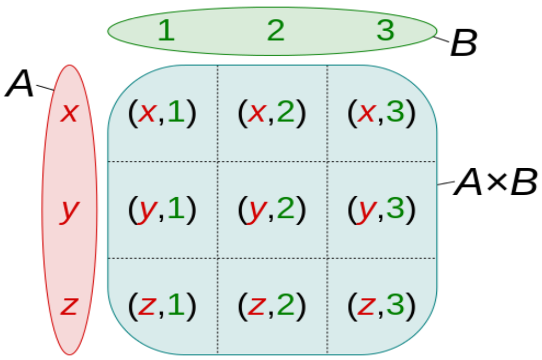
Laws of Algebra of Sets:

Commutative laws: $A \cup B = B \cup A$ $A \cap B = B \cap A$	Associative laws: $(A \cup B) \cup C = A \cup (B \cup C)$ $(A \cap B) \cap C = A \cap (B \cap C)$
Distributive laws: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Identity laws: $A \cup \varnothing = A$ $A \cap U = A$
Complement laws: $A \cup A^C = U$ $A \cap A^C = \varnothing$	Idempotent laws: $A \cup A = A$ $A \cap A = A$
Domination laws: * $A \cup U = U$ $A \cap \varnothing = \varnothing$ ** Note: In some books, it is said as Identity laws.*	Absorption laws: $A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$
De Morgan's laws: $(A \cup B)^C = A^C \cap B^C$ $(A \cap B)^C = A^C \cup B^C$	Double complement or Involution law: $(A^C)^C = A$
(No Name): $A - B = A \cap B^c$	

Ordered pair:

(a, b) is said to be ordered pair if the order in which the objects appear in the pair is significant. i.e. $(a, b) \neq (b, a)$ unless $a = b$

Cartesian product of two sets:



Given two sets *A*, *B*; their Cartesian product $A \times B$ is the set of all ordered pairs (a, b) such that $a \in A$ & $b \in B$. Mathematically, $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$

Let, $A = \{x, y, z\}$ and $B = \{1,2,3\}$, then

$A \times B = \{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3), (z, 1), (z, 2), (z, 3)\}$

$B \times A = \{(1, x), (1, y), (1, z), (2, x), (2, y), (2, z), (3, x), (3, y), (3, z)\}$

$\because (x, 1) \neq (1, x)$ So, $A \times B \neq B \times A$ i.e. Cartesian product of two sets is NOT commutative.

Summary of Set operations:

<table>
<tr><td>$x \in A \cup B$</td><td colspan="2">$x \in A \ or \ x \in B$</td></tr>
<tr><td>$x \notin A \cup B$</td><td colspan="2">$x \notin A \ and \ x \notin B$</td></tr>
<tr><td>$x \in A \cap B$</td><td colspan="2">$x \in A \ and \ x \in B$</td></tr>
<tr><td>$x \notin A \cap B$</td><td colspan="2">$x \notin A \ or \ x \notin B$</td></tr>
<tr><td>$x \notin B$</td><td colspan="2">$x \in B^c$</td></tr>
<tr><td>$x \in B^c$</td><td colspan="2">$x \notin B$</td></tr>
<tr><td>$x \in A - B$</td><td colspan="2">$x \in A \ and \ x \notin B$
$\Rightarrow x \in A \ and \ x \in B^c$
$\Longrightarrow x \in A \cap B^c$</td></tr>
<tr><td>$x \notin A - B$</td><td colspan="2">$x \notin A \ or \ x \in B$</td></tr>
<tr><td>$A \subseteq B$</td><td colspan="2">Let $x \in A$ be any element. If $x \in A \Rightarrow x \in B$, then we say that $A \subseteq B$</td></tr>
<tr><td rowspan="2">$A = B$</td><td>$A \subseteq B$</td><td>$x \in A \Rightarrow x \in B \quad (\forall \, x \in A)$</td></tr>
<tr><td>$B \subseteq A$</td><td>$y \in B \Rightarrow y \in A \quad (\forall \, y \in B)$</td></tr>
</table>

State & Prove De-Morgan's law:

If A & B be any two sets, then $(A \cap B)^c = A^c \cup B^c$ and $(A \cup B)^c = A^c \cap B^c$

Proof:

Let $x \in (A \cap B)^c$ be any element. $x \in (A \cap B)^c$ $\Longrightarrow x \notin (A \cap B)$ $\Longrightarrow x \notin A \; or \; x \notin B$ $\Longrightarrow x \in A^c \; or \; x \in B^c$ $\Longrightarrow x \in A^c \cup B^c$ $\therefore (A \cap B)^c \subseteq A^c \cup B^c$	Let $y \in A^c \cup B^c$ be any element. $y \in A^c \cup B^c$ $\Longrightarrow y \in A^c \; or \; y \in B^c$ $\Longrightarrow y \notin A \; or \; y \notin B$ $\Longrightarrow y \notin A \cap B$ $\Longrightarrow y \in (A \cap B)^c$ $\therefore A^c \cup B^c \subseteq (A \cap B)^c$
$\therefore (A \cap B)^c = A^c \cup B^c$	

Let $x \in (A \cup B)^c$ be any element. $x \in (A \cup B)^c$ $\Longrightarrow x \notin (A \cup B)$ $\Longrightarrow x \notin A \; and \; x \notin B$ $\Longrightarrow x \in A^c \; and \; x \in B^c$ $\Longrightarrow x \in A^c \cap B^c$ $\therefore (A \cup B)^c \subseteq A^c \cap B^c$	Let $y \in A^c \cap B^c$ be any element. $y \in A^c \cap B^c$ $\Longrightarrow y \in A^c \; and \; y \in B^c$ $\Longrightarrow y \notin A \; and \; y \notin B$ $\Longrightarrow y \notin A \cup B$ $\Longrightarrow y \in (A \cup B)^c$ $\therefore A^c \cap B^c \subseteq (A \cup B)^c$
$\therefore (A \cup B)^c = A^c \cap B^c$	

Number of members of set:

Let A, B, C be any three sets. Then

$n(A \cup B)$

$= n(A) + n(B) - n(A \cap B)$

$n(A \cup B \cup C)$

$= n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$

If A & B are disjoint sets, then $A \cap B = \phi \ \therefore n(A \cap B) = 0$

$$\therefore n(A \cup B) = n(A) + n(B)$$

Using the above, we can also say that,

- $n(A) = n(A - B) + n(AB)$
- $n(A \cap B^c) = n(A - B) = n(A) - n(A \cap B)$
- $n(A \cup B) = n(A - B) + n(AB) + n(B - A)$
- $n(A) + n(A^c) = n(U)$
- $n\{(A \cup B)\} + n\{(A \cup B)^c\} = n(U) \ \Rightarrow \ n(A^c \cap B^c) = n(U) - n(A \cup B)$
- $n(A^c \cup B^c) = n(U) - n(A \cap B)$