RELATION

<u>Definition of relation</u>:

A relation between two non-empty sets is a collection of ordered pairs, containing one object from each set.

Let $\mathcal{R}$ be a relation from set A to set B. Then $\mathcal{R} = \{(x, y) \mid x \in A \text{ } and \text{ } y \in B\}$

If $\mathcal{R}$ be a relation from set A to set B and $(x, y) \in \mathcal{R}$ [where $x \in A \text{ } and \text{ } y \in B$]

then we can write this as $x \, \mathcal{R} \, y$. From the definition of relation, we can easily guess that $\mathcal{R} \subseteq A \times B$

Note: If it is said that $\mathcal{R}$ be a relation on set A only, we pre-assume that $\mathcal{R} \subseteq A \times A$

A relation from set A to set B is any subset of $A \times B$. Let $n(A) = p$ and $n(B) = q$, then $n(A \times B) = pq$. Then maximum number of possible relations from set A to set B = number of subsets of $A \times B$ = 2^{pq}

<u>Example – 1</u>: Let $A = \{1, 2, 3, 4\}$, $B = \{1, 2, 3, 5\}$ and $\mathcal{R}$ is a relation defined from set A to set B where $(x, y) \in \mathcal{R} \Rightarrow x > y$. Find $\mathcal{R}$

Solution: $A \times B = \begin{Bmatrix} (1,1), (1,2), (1,3), (1,5), (2,1), (2,2), (2,3), (2,5), \\ (3,1), (3,2), (3,3), (3,5), (4,1), (4,2), (4,3), (4,5) \end{Bmatrix}$

As, $(x, y) \in \mathcal{R} \Rightarrow x > y$ and $\mathcal{R} \subseteq A \times B$

So, $\mathcal{R} = \{(2,1), ((3,1), (3,2), (4,1), (4,2), (4,3)\}$

Domain & Range of Relation:

Domain: If $\mathcal{R}$ be a relation from set A to set B, then domain of $\mathcal{R}$ is defined by the set $\{x \mid (x, y) \in \mathcal{R}\}$

Range: If $\mathcal{R}$ be a relation from set A to set B, then range of $\mathcal{R}$ is defined is defined by the set $\{y \mid (x, y) \in \mathcal{R}\}$

In our above example 1, domain of $\mathcal{R} = \{2, 3, 4\}$ and range of $\mathcal{R} = \{1, 2, 3\}$

<u>Example – 2</u>: Let $A = \{2, 3, 4, 5\}$, $B = \{6, 8, 10, 30\}$ and $\mathcal{R}$ is a relation defined from set A to set B where $(x, y) \in \mathcal{R} \Rightarrow x$ is relatively prime to y. Find $\mathcal{R}$ and its domain, range.

Solution: $\mathcal{R} = \{(3,8), (3,10), (5,6), (5,8)\}$. Domain of $\mathcal{R} = \{3, 5\}$ and range of $\mathcal{R} = \{6, 8, 10\}$

Different types of Relation:

- *<u>Inverse Relation</u>*: Let ρ be a relation from set A to set B. Every such relation has an inverse relation, which is denoted by ρ^{-1} . Then ρ^{-1} is a relation from set B to set A. If $\rho = \{(x, y) \mid x \in A \; and \; y \in B\}$, then $\rho^{-1} = \{(y, x) \mid (x, y) \in \rho\}$. In example 2, $\mathcal{R}^{-1} = \{(8,3), (10,3), (6,5), (8,5)\}$. We see that domain of $\mathcal{R}$ = range of $\mathcal{R}^{-1}$ and range of $\mathcal{R}$ = domain of $\mathcal{R}^{-1}$
- *<u>Empty Relation</u>*: Let ρ be a relation from set A to set B. If $\rho = \phi$ (i.e. null set), then the relation is called an empty relation. E.g. let $A = \{1, 3\}$ and $B = \{5, 7\}$. Then $\rho = \{(a, b) \mid a \in A, b \in B \;\&\; a \times b \; is \; an \; even \; number\}$ is an empty relation from A to B.
- *<u>Universal Relation:</u>* Let ρ be a relation from set A to set B. If $\rho = A \times B$, then the relation is called a universal relation. E.g. let $A = \{1, 3\}$ and $B = \{5, 7\}$. Then $\rho = \{(1,5), (1,7), (3,5), (3,7)\}$ is a universal relation from set A to B.
- *<u>Identity Relation</u>*: Let A be a non-empty set. An identity relation on set A is denoted by I_A and is defined by $I_A = \{(a, b) : a = b \;\; \forall a, b \in A\}$. E.g. in set $A = \{1, 2, 3\}$ then $\rho = \{(1,1), (2,2), (3,3)\}$ is an identity relation, but $\rho = \{(1,1), (2,2)\}$ isn't an identity relation because $(3,3) \notin \rho$. Again $\rho_1 = \{(1,1), (2,2), (1,2)\}, (3,3)\}$ isn't an identity relation because $(1,2) \in \rho_1$

- *Reflexive Relation*: Let A be a non-empty set. A relation ρ defined on a set A, is called a reflexive relation, if $(a, a) \in \rho \;\; \forall a \in A$

  E.g. let $A = \{1, 2, 3\}$ and three relations defined on this set are following –

  $R_1 = \{(1,1), (2,2), (3,3)\}$, $R_2 = \{(1,1), (2,2), (3,3), (1,2)\}$, $R_3 = \{(1,1), (2,2)\}$

  Here R_1 is both identity & reflexive relation. R_2 is reflexive but not an identity relation as $(1,2) \in R_2$. R_3 is neither reflexive nor identity relation as $(3,3) \notin R_3$. We see that all identity relation is reflexive but all reflexive relation is not an identity relation.

- *Symmetric Relation*: Let A be a non-empty set. A relation ρ defined on a set A, is called a symmetric relation, if $(a, b) \in \rho \;\Rightarrow\; (b, a) \in \rho$ when $a, b \in A$

  E.g. let $A = \{1, 2, 3\}$ and three relations defined on this set are following –

  $R_1 = \{(1,1), (2,2), (3,3)\}$,

  $R_2 = \{(1,1), (2,2), (2,1), (1,2)\}$,

  $R_3 = \{(1,1), (2,2), (1,2), (2,1), (1,3)\}$

  R_1 is both identity & symmetric relation.

  R_2 is symmetric but not an identity [as, $(1,2), (2,1) \in R_2$] and reflexive relation [as $(3,3) \notin R_2$]

  R_3 is not symmetric as $(3,1) \notin R_3$

  Note: (i) All identity & universal relations in a non-empty set A are always symmetric.

  (ii) Let ρ be a relation in a non-empty set A. ρ is symmetric if and only if $\rho = \rho^{-1}$

- *Transitive Relation*: Let A be a non-empty set. A relation ρ defined on a set A, is called a transitive relation, if and only if $a\rho b$ and $b\rho c$ implies $a\rho c$ [i.e. $(a, b) \in \rho$ and $(b, c) \in \rho \Rightarrow (a, c) \in \rho$]

  E.g. Let ρ be a relation in set $\mathbb{R}$ defined by $\rho = \{(x, y): x > y \ \forall x, y \in \mathbb{R}\}$ is a transitive relation.

  E.g. let $A = \{1, 2, 3\}$ and three relations defined on this set are following –

  $R_1 = \{(1,1), (2,2), (3,3)\}$,

  $R_2 = \{(1,1), (2,2), (3,3), (1,2)\}$,

  $R_3 = \{(1,1), (2,1), (1,2)\}$

  Here R_1 is identity, reflexive, symmetric & transitive relation.

  R_2 is reflexive, transitive, but not identity [as, $(1,2) \in R_2$], not symmetric [as $(2,1) \notin R_2$]

  R_3 is not transitive, as $(2,1) \in R_3$, $(1,2) \in R_3$ but $(2,2) \notin R_3$

- *Equivalence Relation*: Let A be a non-empty set. A relation ρ defined on a set A, is called an equivalence relation, if ρ is reflexive, symmetric & transitive.

  E.g. let L be a set of all straight lines drawn on a plane and a relation ρ in set L is defined by $l_1 \rho l_2 \Rightarrow l_1 \parallel l_2$. Here ρ is an equivalence relation. All identity relations are equivalence relations.

Congruence modulo m: Let $\mathbb{Z}$ be a set of all integers and m be a fixed positive integer. If $a, b \in \mathbb{Z}$ then a, b is said to be $congruence\ modulo\ m$, if $a - b$ is divisible by m and it is denoted by $a \equiv b$ (mod m).

So, $a \equiv b$ (mod m) $\Rightarrow a - b$ is divisible by m

E.g. $22 \equiv 1$ (mod 7) as, $22 - 1 = 21$ is divisible by 7

But, $22 \not\equiv 1$ (mod 5) as, $22 - 1 = 21$ is NOT divisible by 5.