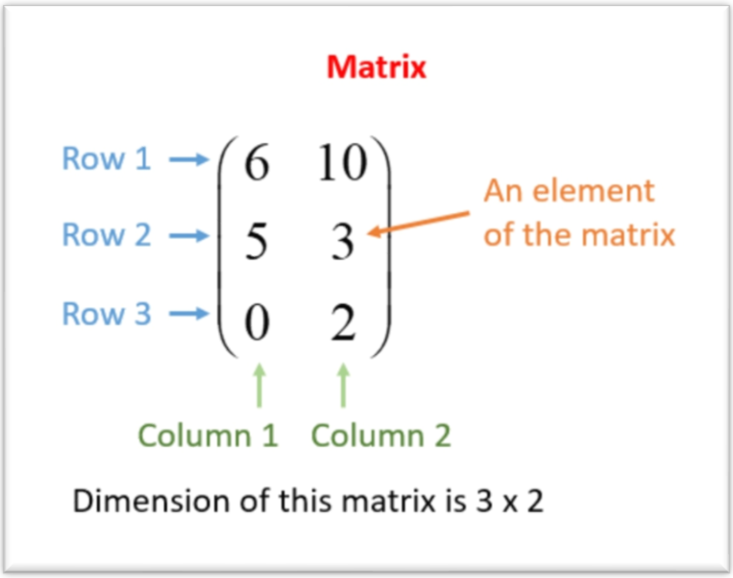
MATRIX

History of Matrix:

The matrix has a long history of application in solving linear equations. It was not the matrix but a certain number associated with a square array of numbers called the determinant that was first recognized. The term "matrix" was coined by James Joseph Sylvester in 1850, but it was his friend mathematician Arthur Cayley who developed algebraic aspects of matrices in same year. An English mathematician named Cullis was the first to use modern bracket notation for matrices in 1913 and demonstrated the first significant use of the notation $A = [a_{ij}]$ to represent a matrix where a_{ij} refers to an element found in i^{th} row and j^{th} column. Matrices can be used to compactly write and work with multiple linear equations, referred to as a system of linear equations. Matrices have wide applications in engineering, physics, economics, and statistics as well as in various branches of mathematics. Matrices also have important applications in computer graphics, where they have been used to represent rotations and other transformations of images.

Basic concept:



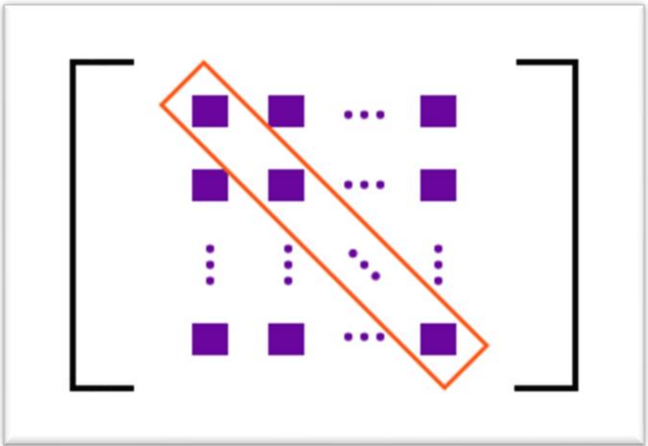
In mathematics, a matrix (plural matrices) is a rectangular array of numbers, symbols, or expressions, arranged in rows and columns. Matrices are commonly written in box brackets like ( ) or [ ]. The horizontal and vertical lines of entries in a matrix are called rows and columns, respectively. The individual item in a matrix is called its member/element/entry. In a common notation, a capital letter denotes a matrix, and the corresponding small letter with a double subscript describes an element of the matrix.

$$\begin{bmatrix} 6 & 4 & 24 \\ 1 & -9 & 8 \end{bmatrix}$$

This is an example of a matrix, which has 2 rows and 3 columns. A matrix with 2 rows and 3 columns is called 2×3 matrix (read as, 2-by-3)

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & \cdots & n \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ \vdots \\ m \end{array} & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

A matrix with m rows and n columns is called an $m \times n$ matrix (read as, m-by-n matrix), "$m \times n$" is also known as order/dimension of matrix. a_{11} denotes a member staying at 1st row & 1st column. In general, a_{ij} denotes a member staying at i^{th} row & j^{th} column.



Principal or main diagonal of a matrix: The diagonal from the top left corner to the bottom right corner of a square matrix is called the main diagonal or principal diagonal of a matrix.

Several types of matrices:

● *Row matrix*: A matrix which has only one row, is known as row matrix. E.g. $(1 \quad 3 \quad 5)$ is a row matrix.

● *Column matrix*: A matrix which has only one column, is known as column matrix. E.g. $\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$ is a column matrix.

● *Square matrix*: A matrix which has the same number of rows and columns is called a square matrix. So, if any square matrix has m number of rows & columns, then order of the said matrix is "$m \times m$" or simply m. E.g. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -3 & 0 & -4 \end{pmatrix}$ are square matrices of order 2, 3 respectively.

● *Zero (null) matrix*: A matrix, whose all elements are 0, is known as zero matrix and is denoted by O (alphabet). E.g. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ are zero matrices.

● *Diagonal matrix*: A square matrix whose off-diagonal elements are 0, is known as diagonal matrix. Here we note that diagonal elements may/may not be equal to zero.

E.g. $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix}$, $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix}$ are diagonal matrices.

● *Scalar matrix*: A diagonal matrix is called a scalar matrix if all the diagonal entries are same and non-zero.

E.g. $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, $\begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$ are scalar matrices (where, $c \neq 0$)

● *Identity (unit) matrix*: A square matrix whose diagonal entries are all equal to 1 and whose off-diagonal entries are all equal to zero, is called an identity matrix. It is denoted by I. E.g.

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are square matrices. Here subscript of I denotes order of the corresponding square matrix.

Equality of two matrices:

Two matrices are equal if they have the same dimension/order and the corresponding elements are identical.

Example: Given that the following matrices are equal, find the values of x, y and z .

$$\begin{pmatrix} x+3 & -1 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 6 & y \\ z-3 & 5 \end{pmatrix}$$

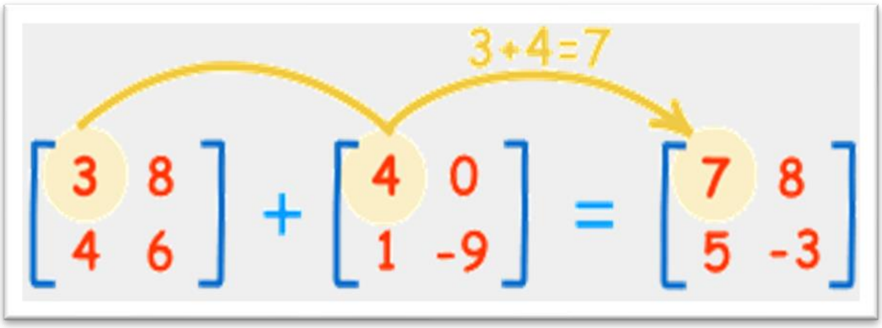
As the matrices are equal, we get $x + 3 = 6 \Rightarrow x = 3$

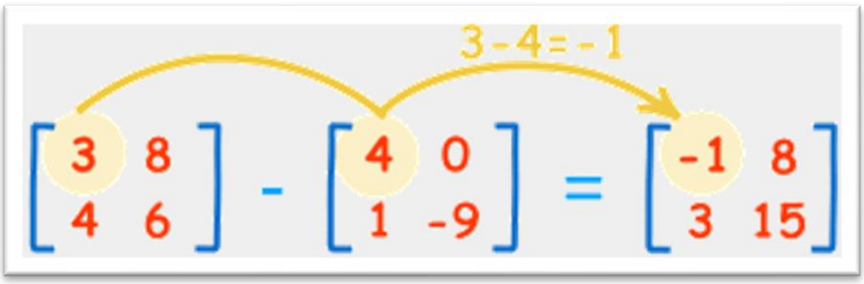
$y = -1,\ z - 3 = 4 \Rightarrow z = 7$.

So, $x = 3,\ y = -1,\ z = 7$

Addition/Subtraction of two/more matrices:

We can add (or, subtract) two or more matrices of *same order* by adding (or, subtracting) the corresponding elements.

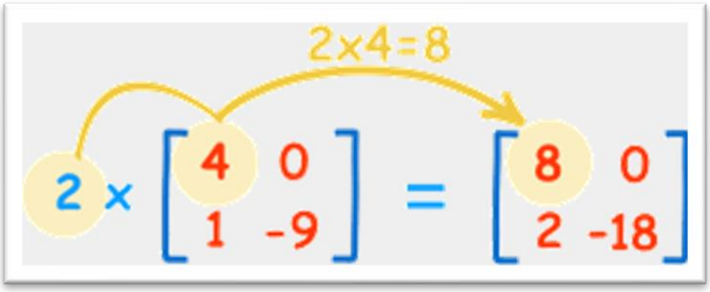




Note: We can't add/subtract matrices, if they are in different order. Matrix addition is commutative and is also associative, so the following is true: $A + B = B + A$,

$(A + B) + C = A + (B + C)$ [Here A, B, C are three matrices of same order.]

Multiplication of a matrix by a scalar quantity:



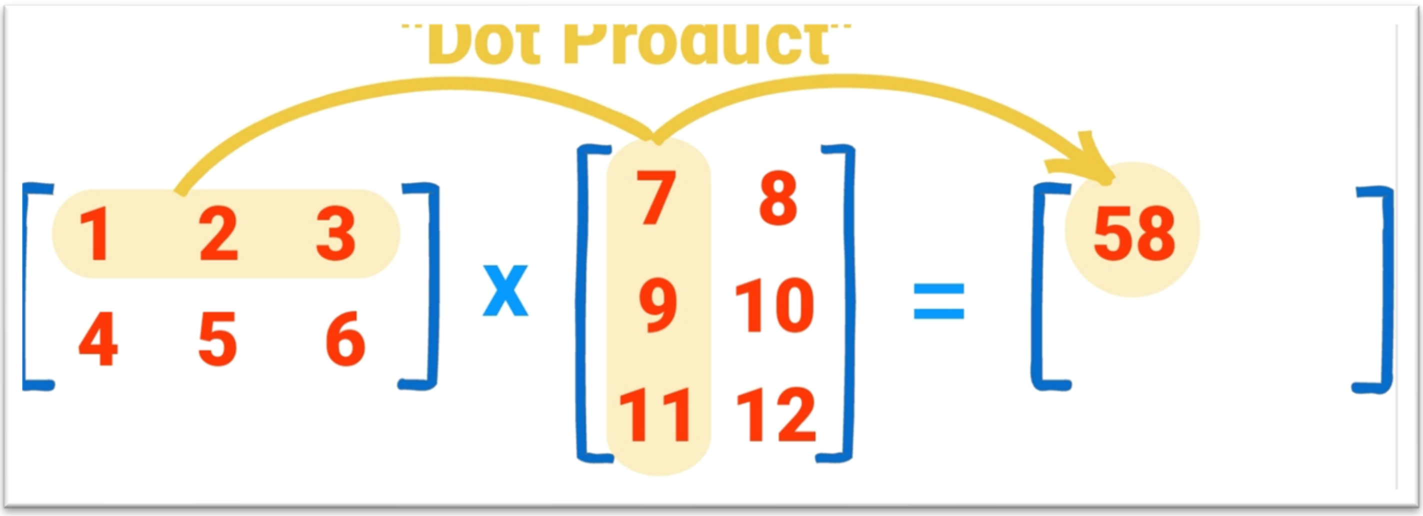
When we multiply a matrix by some scalar quantity (e.g. 2, 3, 1/2 etc), then we have to multiply all members of the matrix by the said scalar to form new matrix.

Multiplication of two matrices:

The rule for matrix multiplication is that two matrices can be multiplied only when the *number of columns of the first matrix* equals to the *number of rows of the second matrix*.

Let A is a matrix of order $m \times n$ and B is a matrix of order $p \times q$. So, $A \cdot B$ can be possible only when $n = p$ and order of product matrix is $m \times q$.

If the above condition is satisfied, then to multiply a matrix by another matrix, we need to do "dot product" between rows of 1st matrix by columns of 2nd matrix.



The "Dot Product" is where we multiply matching members, then sum up:

$(1, 2, 3) \cdot (7, 9, 11) = 1 \times 7 + 2 \times 9 + 3 \times 11 = 58$

Now we do dot product again between 1st row (of 1st matrix) and 2nd column (of 2nd matrix): $(1, 2, 3) \cdot (8, 10, 12) = 1 \times 8 + 2 \times 10 + 3 \times 12 = 64$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ & \end{bmatrix}$$

We can do the same thing for the 2nd row and 1st column:

$$(4, 5, 6) \cdot (7, 9, 11) = 4 \times 7 + 5 \times 9 + 6 \times 11 = 139$$

And for the 2nd row and 2nd column:

$$(4, 5, 6) \cdot (8, 10, 12) = 4 \times 8 + 5 \times 10 + 6 \times 12 = 154$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix}$$

Some more examples of matrix multiplication:

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \times 4 + 2 \times 5 + 3 \times 6 \end{bmatrix} = \begin{bmatrix} 32 \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 4\times1 & 4\times2 & 4\times3 \\ 5\times1 & 5\times2 & 5\times3 \\ 6\times1 & 6\times2 & 6\times3 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 12 \\ 5 & 10 & 15 \\ 6 & 12 & 18 \end{bmatrix}$$

- From the above example, we come to know that matrix multiplication is not commutative (in general). So, if A & B are two matrices, then $AB \neq BA$;
- As matrix multiplication is not commutative (in general), so algebraic identities such as $(A \pm B)^2 = A^2 \pm 2AB + B^2$, $A^2 - B^2 = (A+B)(A-B)$, etc.. do not hold for matrix.
- However, matrix multiplication follows associative law. Let A, B, C are three matrices, then $A(BC) = (AB)C$
- Matrix addition also follows distributive law, i.e. $A(B + C) = AB + AC$, $(B + C)A = BA + CA$
- If A and I be any two matrices of same order, then $A \times I = I \times A = A$, where I is an identity matrix.
- If A and B are two non-zero matrices (a matrix which has at least one non-zero element, is called non-zero matrix), then it is possible that their product is a zero matrix. E.g. $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$. But $AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is a zero matrix.
- Let A be any square matrix. The matrix A is called *idempotent matrix*, if $A^2 = A$
- Let A be any square matrix. The matrix A is called *nilpotent matrix* if there is any non-negative integer k such that A^k is a zero matrix. The smallest such an integer k is called degree/index of matrix A.

Transpose of a matrix:

$$\begin{pmatrix} 5 & 4 \\ 4 & 0 \\ 7 & 10 \\ -1 & 8 \end{pmatrix}^T_{4\times 2} = \begin{pmatrix} 5 & 4 & 7 & -1 \\ 4 & 0 & 10 & 8 \end{pmatrix}_{2\times 4}$$

Transpose of a matrix is obtained by changing rows to columns or columns to rows. Transpose of a matrix A is denoted by A' or A^T.

Properties of transposes: For any matrices (A, B, C) (assume that they are compatible for addition & multiplication),

- $(A^T)^T = A$
- $(k \cdot A)^T = k \cdot A^T$ (where k is a scalar)
- $(A + B)^T = A^T + B^T$, similarly, $(A + B + C)^T = A^T + B^T + C^T$ and so on …
- $(AB)^T = B^T A^T$, similarly, $(ABC)^T = C^T B^T A^T$ and so on …
- $(A^n)^T = (A^T)^n$ where $n \in \mathbb{N}$

Symmetric & skew-symmetric matrix:

A square matrix $A = [a_{ij}]_{m\times m}$ is called a *symmetric matrix*, if $a_{ij} = a_{ji}$ for all values of i, j

E.g. $A = [a_{ij}]_{3\times 3} = \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}$. Here, $a_{12} = a_{21} = h,\ a_{13} = a_{31} = g,\ a_{23} = a_{32} = f$

A square matrix $A = [a_{ij}]_{m\times m}$ is called a *skew-symmetric matrix*, if $a_{ij} = -a_{ji}$ for all values of i, j

E.g. $A = [a_{ij}]_{3\times 3} = \begin{pmatrix} 0 & h & -g \\ -h & 0 & f \\ g & f- & 0 \end{pmatrix}$

Properties –

- If A is a symmetric matrix, then $A^T = A$
- If A is a skew-symmetric matrix, then $A^T = -A$
- For skew-symmetric matrix, $a_{ij} = -a_{ji}$ (for all i, j) $\Rightarrow a_{ii} = -a_{ii}$ (for $i = j$) $\Rightarrow 2a_{ii} = 0 \Rightarrow a_{ii} = 0$. So, all diagonal elements of a skew-symmetric matrix are always 0.

- Every square matrix can be expressed as a sum of a symmetric matrix and a skew-symmetric matrix. Let A be any square matrix, then $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$. It can be easily proved that $\frac{1}{2}(A + A^T)$ is a symmetric matrix and $\frac{1}{2}(A - A^T)$ is a skew-symmetric matrix.

Basic concept about determinant:

The determinant is a *scalar value* that can be computed from the elements of a square matrix. The determinant of a *square matrix* A is denoted by $\det(A)$ or $|A|$

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then its determinant is $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Minor & Cofactor of a determinant:

$$\Delta = \begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix}$$

Take a 2×2 determinant. We have elements $a_{11} = 3,\ a_{12} = 2,\ a_{21} = 1,\ \ a_{22} = 4$. Let minors are $M_{11},\ M_{12},\ M_{21},\ M_{22}$. From the following figure, try to understand that how its values are obtained.

$$M_{11} = \begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} = 4$$

$$M_{12} = \begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} = 1$$

$$M_{21} = \begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} = 2$$

$$M_{22} = \begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} = 3$$

We imagine a horizontal and a vertical line through the element, whose minor is required. Thereafter we imagine a matrix with the remaining elements, and its determinant is the required minor of that element.

Now we need cofactors. Let cofactors are $A_{11},\ A_{12},\ A_{21},\ A_{22}$.

Formula: $A_{ij} = (-1)^{i+j} \cdot M_{ij}$

$$
\begin{aligned}
A_{11} &= (-1)^{1+1} M_{11} \\
&= (-1)^{2} M_{11} \\
&= M_{11} \\
&= 4
\end{aligned}
$$

$$
\begin{aligned}
A_{21} &= (-1)^{2+1} M_{21} \\
&= (-1)^{3} M_{21} \\
&= -1 \times M_{21} \\
&= -1 \times 2 \\
&= -2
\end{aligned}
$$

$$
\begin{aligned}
A_{12} &= (-1)^{1+2} M_{12} \\
&= (-1)^{3} M_{12} \\
&= -1 \times M_{12} \\
&= -1 \times 1 \\
&= -1
\end{aligned}
$$

$$
\begin{aligned}
A_{22} &= (-1)^{2+2} M_{22} \\
&= (-1)^{4} M_{22} \\
&= 1 \times M_{22} \\
&= 3
\end{aligned}
$$

Tips of easy remembering of sign: $\begin{pmatrix} + & - \\ - & + \end{pmatrix}$

Now we take a square matrix of order 3 and find out cofactors of each element.

For a 3 × 3 matrix

$$\Delta = \begin{vmatrix} 9 & 2 & 1 \\ 5 & -1 & 6 \\ 4 & 0 & -2 \end{vmatrix}$$

We have elements,

$a_{11} = 9$	$a_{21} = 5$	$a_{31} = 4$
$a_{12} = 2$	$a_{22} = -1$	$a_{32} = 0$
$a_{13} = 1$	$a_{23} = 6$	$a_{33} = -2$

Minors will be: $M_{11},\ M_{12},\ M_{13},\ M_{21},\ M_{22},\ M_{23},\ M_{31},\ M_{32},\ M_{33}$

$$M_{11} = \begin{vmatrix} 9 & 2 & 1 \\ 5 & -1 & 6 \\ 4 & 0 & -2 \end{vmatrix} = \begin{vmatrix} -1 & 6 \\ 0 & -2 \end{vmatrix} = (-1) \times (-2) - 0 \times 6 = 2 - 0 = 2$$

$$M_{12} = \begin{vmatrix} 9 & 2 & 1 \\ 5 & -1 & 6 \\ 4 & 0 & -2 \end{vmatrix} = \begin{vmatrix} 5 & 6 \\ 4 & -2 \end{vmatrix} = 5 \times (-2) - 4 \times 6 = -10 - 24 = -34$$

$$M_{13} = \begin{vmatrix} 9 & 2 & 1 \\ 5 & -1 & 6 \\ 4 & 0 & -2 \end{vmatrix} = \begin{vmatrix} 5 & -1 \\ 4 & 0 \end{vmatrix} = 5 \times 0 - 4 \times (-1) = 0 + 4 = 4$$

$$M_{21} = \begin{vmatrix} 9 & 2 & 1 \\ 5 & -1 & 6 \\ 4 & 0 & -2 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 0 & -2 \end{vmatrix} = 2 \times (-2) - 0 \times 1 = -4 - 0 = -4$$

$$M_{22} = \begin{vmatrix} 9 & 2 & 1 \\ 5 & 1 & 6 \\ 4 & 0 & -2 \end{vmatrix} = \begin{vmatrix} 9 & 1 \\ 4 & -2 \end{vmatrix} = 9 \times (-2) - 4 \times 1 = -18 - 4 = -22$$

$$M_{23} = \begin{vmatrix} 9 & 2 & 1 \\ 5 & 1 & 6 \\ 4 & 0 & -2 \end{vmatrix} = \begin{vmatrix} 9 & 2 \\ 4 & 0 \end{vmatrix} = 9 \times 0 - 4 \times 2 = 0 - 8 = -8$$

$$M_{31} = \begin{vmatrix} 9 & 2 & 1 \\ 5 & -1 & 6 \\ 4 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ -1 & 6 \end{vmatrix} = 2 \times 6 - (-1) \times 1 = 12 + 1 = 13$$

$$M_{32} = \begin{vmatrix} 9 & 2 & 1 \\ 5 & -1 & 6 \\ 4 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 9 & 1 \\ 5 & 6 \end{vmatrix} = 9 \times 6 - 5 \times 1 = 54 - 5 = 49$$

$$M_{33} = \begin{vmatrix} 9 & 2 & 1 \\ 5 & -1 & 6 \\ 4 & 0 & -2 \end{vmatrix} = \begin{vmatrix} 9 & 2 \\ 5 & -1 \end{vmatrix} = 9 \times (-1) - 5 \times 2 = -9 - 10 = -19$$

Co-factors will be: $A_{11}, A_{12}, A_{13}, A_{21}, A_{22}, A_{23}, A_{31}, A_{32}, A_{33}$

$$A_{11} = (-1)^{1+1} M_{11} = M_{11} = 2$$

$$A_{12} = (-1)^{1+2} M_{12} = -1 \times M_{12} = -1 \times -34 = 34$$

$$A_{13} = (-1)^{1+3} M_{13} = M_{13} = 4$$

$$A_{21} = (-1)^{2+1} M_{21} = -1 \times M_{21} = -1 \times -4 = 4$$

$$A_{22} = (-1)^{2+2} M_{22} = M_{22} = -22$$

$$A_{23} = (-1)^{2+3} M_{23} = -1 \times M_{23} = -1 \times -8 = 8$$

$$A_{31} = (-1)^{3+1} M_{31} = M_{31} = 13$$

$$A_{32} = (-1)^{3+2} M_{32} = -1 \times M_{32} = -1 \times 49 = -49$$

$$A_{33} = (-1)^{3+3} M_{33} = M_{33} = -19$$

Easy way for remembering sign: $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$ (Note that starting with $+$, sign alters horizontally/vertically). They are the sign of co-factors, that we have to add with minors.)

<u>Finding the value of a determinant</u>:

Name of the method: expansion method

Let, $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

How to find its value: we take any one row/column and then do “dot product” of the elements of selected row/column with its cofactor. See below:

$|A| = a_{11} \cdot A_{11} + a_{12} \cdot A_{12} + a_{13} \cdot A_{13} = a_{12} \cdot A_{12} + a_{22} \cdot A_{22} + a_{32} \cdot A_{32} = \cdots$

(we get same value for any row/column)

Adjoint of a matrix:

The adjoint of a matrix is defined as the transpose of the cofactor matrix of that particular matrix. For a matrix A, the adjoint is denoted as $adj(A)$.

E.g., if $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then $adj(A) = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}^T$

Adjoint matrix is used to solve a system of linear equations.

Inverse of a square matrix:

Let A be any square matrix of order n. If there is another square matrix B of order n, such that $A \cdot B = B \cdot A = I_n$, then the matrix B is called inverse of matrix A and is denoted by $B = A^{-1}$

Properties of inverse matrix (assume A, B are square matrices of order n) –

- $A \cdot A^{-1} = A^{-1} \cdot A = I_n$
- $(A^{-1})^{-1} = A$
- $A \cdot adj(A) = |A| \cdot I \;\Rightarrow\; A^{-1} = \frac{adj(A)}{|A|}$ (So, A^{-1} exists, if $|A| \neq 0$)
- If a square matrix has inverse, then it is unique.
- $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$
- $(A^T)^{-1} = (A^{-1})^T$ (assuming that A^{-1} exists)
- $|adj(A)| = |A|^{n-1}$, when $|A| \neq 0$
- $adj(AB) = adj(B) \cdot adj(A)$
- If matrix A is symmetric, then $adj(A)$ is also symmetric.
- If $|A| \neq 0$, then $|A^{-1}| = \frac{1}{|A|}$

Singular & Non-singular matrix:

A square matrix A is called singular matrix, if $|A| = 0$

A square matrix A is called non-singular matrix, if $|A| \neq 0$

Involutory matrix:

A square matrix A is called an involutory matrix, if $A = A^{-1}$

Orthogonal matrix:

A square matrix A is called an orthogonal matrix, if $A \cdot A^T = A^T \cdot A = I$

Properties –

- If A is an orthogonal matrix, then $|A| = \pm 1$. Now if $|A| = 1$ then matrix A is called *proper* orthogonal matrix, else A is called an *improper* orthogonal matrix.
- If A is an orthogonal matrix, then $A^{-1} = A^T$

Solving system of linear equation with matrices:

Let's take a system of linear equations with two variables x and y.

$$\left.\begin{matrix} a_{11}x + a_{12}y = b_1 \\ a_{21}x + a_{22}y = b_2 \end{matrix}\right\} \cdots\cdots (i)$$

Assume, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

So, $AX = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{pmatrix}$

Then we can write equations (i) as: $\quad AX = B$

Now, $AX = B$

$\Rightarrow A^{-1}(AX) = A^{-1}B$ (multiplying both sides by A^{-1})

$\Rightarrow (A^{-1}A)X = A^{-1}B$ (matrix multiplication obeys associative law)

$\Rightarrow I \cdot X = A^{-1}B$ (as, $A^{-1}A = I$)

$\Rightarrow X = A^{-1}B$ (as, $I \times X = X$)

Now, we take system of linear equations with three variables x, y, z

$$x + y + z = 6$$

$$2y + 5z = -4$$

$$2x + 5y - z = 27$$

Let, $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 5 \\ 2 & 5 & -1 \end{pmatrix}$; $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $B = \begin{pmatrix} 6 \\ -4 \\ 27 \end{pmatrix}$

So, we write the provides equations as: $AX = B \Rightarrow X = A^{-1}B$

Now, $A^{-1} = \frac{adj(A)}{|A|}$

$$adj(A) = \begin{pmatrix} \begin{vmatrix} 2 & 5 \\ 5 & -1 \end{vmatrix} & -\begin{vmatrix} 0 & 5 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ 2 & 5 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 5 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & 5 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 2 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 0 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} \end{pmatrix}^T$$

$$= \begin{pmatrix} -27 & 10 & -4 \\ 6 & -3 & -3 \\ 3 & -5 & 2 \end{pmatrix}^T = \begin{pmatrix} -27 & 6 & 3 \\ 10 & -3 & -5 \\ -4 & -3 & 2 \end{pmatrix}$$

$|A| = 1 \times (-27) + 2 \times 3 = -27 + 6 = -21$ (expanding using 1st column, i.e. by taking dot product between members of 1st column & its cofactors)

So, $A^{-1} = \frac{1}{-21}\begin{pmatrix} -27 & 6 & 3 \\ 10 & -3 & -5 \\ -4 & -3 & 2 \end{pmatrix}$

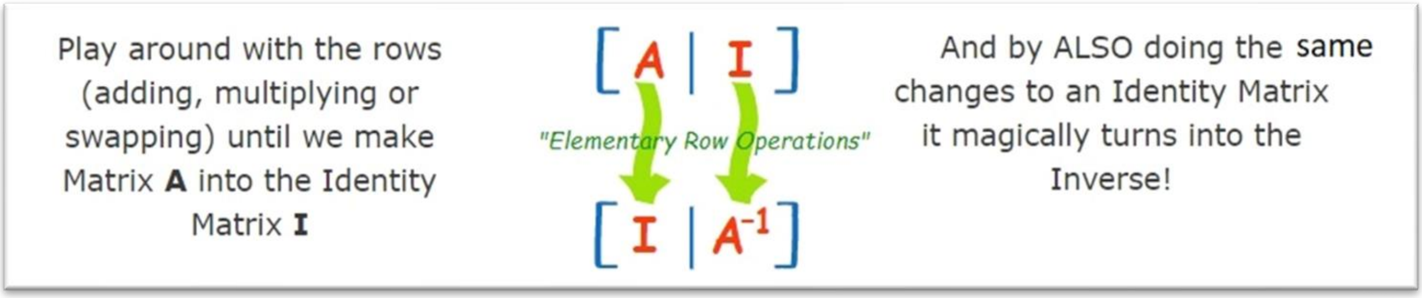
Now, $X = \frac{1}{-21}\begin{pmatrix} -27 & 6 & 3 \\ 10 & -3 & -5 \\ -4 & -3 & 2 \end{pmatrix}\begin{pmatrix} 6 \\ -4 \\ 27 \end{pmatrix} = \frac{1}{-21}\begin{pmatrix} -105 \\ -63 \\ 42 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix}$

So, the solution is: $x = 5,\ y = 3,\ z = -2$

Condition for the number of solutions of $AX = B$

- If $|A| \neq 0$, then unique solution.
- If $|A| = 0$ and $adj(A) \cdot B = 0$, then infinitely many solutions
- If $|A| = 0$ and $adj(A) \cdot B \neq 0$, then no solution/inconsistent

Method of finding the inverse of a non-singular matrix by Elementary Row (or, Column) Transformation: (also called Gauss-Jordan method)



We find the inverse of the matrix, $A = \begin{pmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}$

We can only do these steps in “Elementary Row Transformations” –

- **Swap** rows
- **Multiply/Divide** each element in a row by a constant
- **Replace a row** by adding/subtracting a multiple of another row to it.

A I

$$\left[\begin{array}{ccc|ccc} 3 & 0 & 2 & 1 & 0 & 0 \\ 2 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 5 & 0 & 0 & 1 & 1 & 0 \\ 2 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$ Add

$[R_1 \to R_1 + R_2]$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.2 & 0.2 & 0 \\ 2 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$ Divide by 5

$\left[R_1 \to \frac{1}{5}R_1\right]$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.2 & 0.2 & 0 \\ 0 & 0 & -2 & -0.4 & 0.6 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$ Subtract x 2

$[R_2 \to R_2 - 2R_1]$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.2 & 0.2 & 0 \\ 0 & 0 & 1 & 0.2 & -0.3 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$ Multiply by $-\frac{1}{2}$

$\left[R_2 \to -\frac{1}{2}R_2\right]$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.2 & 0.2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0.2 & -0.3 & 0 \end{array}\right]$$ Swap

$[R_2 \leftrightarrow R_3]$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.2 & 0.2 & 0 \\ 0 & 1 & 0 & -0.2 & 0.3 & 1 \\ 0 & 0 & 1 & 0.2 & -0.3 & 0 \end{array}\right]$$ Subtract

$[R_2 \to R_2 - R_3]$

I A^{-1}

So, $A^{-1} = \begin{pmatrix} 0.2 & 0.2 & 0 \\ -0.2 & 0.3 & 1 \\ 0.2 & -0.3 & 0 \end{pmatrix}$

Similarly, we can use elementary column transformations to find inverse matrix.

I = identity matrix, O = zero matrix, A^T= transpose of A,

1) Form a 2×3 matrix whose element is given by $a_{ij} = \frac{1}{2}(i-2j)^2$

2) Find the value of x, y, z, t such that the matrices $\begin{bmatrix} x+y & y-z \\ 5-t & 7+x \end{bmatrix}$ and $\begin{bmatrix} t-x & z-t \\ z-y & x+z+t \end{bmatrix}$ are equal.

3) Find the value of a, b, c, d such that the matrices $\begin{pmatrix} b+c & c+a \\ 7-d & 6-c \end{pmatrix}$ and $\begin{pmatrix} 9-d & 8-d \\ a+b & a+b \end{pmatrix}$ are equal.

4) If $A = \begin{pmatrix} 1 & 0 \\ -1 & 7 \end{pmatrix}$ then find the value of k such that $A^2 = 8A + kI$

5) If $A = \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}$ (where ω is imaginary cube root of 1), find A^{100}

6) If $A = \begin{pmatrix} 0 & 7 \\ 0 & 0 \end{pmatrix}$ and $f(x) = 1 + x + x^2 + \cdots + x^{20}$, then find $f(A)$.

7) If $2A - 3B = \begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix}$ and $3A + 2B = \begin{pmatrix} -1 & 2 \\ 0 & 4 \end{pmatrix}$, then find the matrices A and B.

8) If $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ then show that $A^2 - 4A + 3I = O$. Hence find A^{-1}.

9) If $A = \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}$, then show that $I + A = (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

10) If $A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ then show that $A^2 - 2A + I_2 = O$; Hence find A^{50}

11) If $A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ then show that $A^2 = 3A - 2I$. Hence show that $A^8 = 255A - 254I$

12) Prove that every square matrix can be expressed as a sum of a symmetric matrix & a skew-symmetric matrix.

13) Express a matrix $\begin{pmatrix} -3 & 4 & 1 \\ 2 & 3 & 0 \\ 1 & 4 & 5 \end{pmatrix}$ as the sum of a symmetric matrix and a skew-symmetric matrix.

14) If $P = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$, then show that $P^2 = P$ and then find a matrix Q such that $3P^2 - 2P + Q = I$

15) Given $A = \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{pmatrix}$ and $f(x) = x^2 - 5x + 6$; Find $f(A)$

16) Show that $A = \begin{pmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{pmatrix}$ is a nilpotent matrix.

17) If $A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$, then show that $B^T(AB)$ is a diagonal matrix.

18) If $\begin{pmatrix} x & 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & 2 \\ 0 & 2 & -4 \end{pmatrix} \begin{pmatrix} x \\ 4 \\ -1 \end{pmatrix} = O$, then find the value of x.

19) Show that the matrix $P = \frac{1}{3}\begin{pmatrix} -1 & 2 & -2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$ is an orthogonal matrix. Hence find P^{-1}

20) If $A = \begin{pmatrix} 6 & 2 & -2 \\ -2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}$ then show that $(A - 2I)(A - 4I) = O$; Hence find A^3

21) If $A \cdot adj(A) = \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}$ then find $|A|$

22) If A is a symmetric matrix, then A^{-1} is – (a) symmetric, (b) skew-symmetric, (c) diagonal, (d) scalar matrix.

23) If A is a symmetric matrix, then A^n is _______________ matrix.

24) If A is a skew-symmetric matrix, then A^n is – (a) symmetric, (b) skew-symmetric, (c) none of these matrices.

25) If A and B are two matrices such that $AB = B$ and $BA = A$. Then $A^2 + B^2$ equals to – (a) $2AB$, (b) AB, (c) $2BA$, (d) $A + B$

26) If A is a square matrix, then $A - A^T$ is – (a) symmetric, (b) skew-symmetric, (c) scalar matrix.

27) If a matrix A is both symmetric and skew-symmetric, then A is – (a) diagonal, (b) identity, (c) zero, (d) none of these matrices.

28) If A, B are two symmetric matrices of same order, then show that – (i) $AB + BA$ is a symmetric matrix, (ii) $AB - BA$ is a skew-symmetric matrix.

29) Show that $B^T AB$ is symmetric/skew-symmetric matrix, if A is symmetric/skew-symmetric matrix.

Form a 2×3 matrix whose element is given by $a_{ij} = \frac{1}{2}(i-2j)^2$

2×3 matrix. $\rightarrow$ [Row = 2, Column = 3]

$$a_{ij} = \frac{1}{2}(i-2j)^2$$

$$\text{Matrix, } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

$$\therefore a_{11} = \frac{1}{2}(1-2)^2 = \frac{1}{2}$$

$$a_{12} = \frac{1}{2}(1-4)^2 = \frac{9}{2}$$

$$a_{13} = \frac{1}{2}(1-6)^2 = \frac{25}{2}$$

$$\therefore a_{21} = \frac{1}{2}(2-2)^2 = 0$$

$$a_{22} = \frac{1}{2}(2-4)^2 = 2$$

$$a_{23} = \frac{1}{2}(2-6)^2 = 8$$

$$\therefore \text{Required matrix is} = \begin{pmatrix} \frac{1}{2} & \frac{9}{2} & \frac{25}{2} \\ 0 & 2 & 8 \end{pmatrix}$$

Find the value of x, y, z, t such that the matrices $\begin{bmatrix} x+y & y-z \\ 5-t & 7+x \end{bmatrix}$ and $\begin{bmatrix} t-x & z-t \\ z-y & x+z+t \end{bmatrix}$ are equal.

$$\begin{pmatrix} x+y & y-z \\ 5-t & 7+x \end{pmatrix} = \begin{pmatrix} t-x & z-t \\ z-y & x+z+t \end{pmatrix}$$

$\therefore\ x+y = t-x \quad \ldots (i)$

$y-z = z-t \quad \ldots (ii)$

$5-t = z-y \quad \ldots (iii)$

$7+x = x+z+t \quad \ldots (iv)$

From (iv), $z+t = 7. \quad \ldots (v)$

From (iii), $y = z+t-5 = 2$

From (ii), $2z-t = 2 \quad (\because y=2) \quad \ldots (vi)$

Solving (v) & (vi), $z=3,\ t=4$

From (i), $2x = t-y = 2 \Rightarrow x = 1$

$\therefore\ x=1,\ y=2,\ z=3,\ t=4.$

Find the value of a, b, c, d such that the matrices $\begin{pmatrix} b+c & c+a \\ 7-d & 6-c \end{pmatrix}$ and $\begin{pmatrix} 9-d & 8-d \\ a+b & a+b \end{pmatrix}$ are equal.

$$\begin{pmatrix} b+c & c+a \\ 7-d & 6-c \end{pmatrix} = \begin{pmatrix} 9-d & 8-d \\ a+b & a+b \end{pmatrix}$$

As the matrices are equal,

$\therefore b+c = 9-d \quad \ldots\ldots (i)$

$c+a = 8-d \quad \ldots\ldots (ii)$

$a+b = 7-d \quad \ldots\ldots (iii)$

$a+b = 6-c \quad \ldots\ldots (iv)$

From (iii) & (iv), $7-d = 6-c \Rightarrow c-d = -1 \quad \ldots\ldots (v)$

Adding (i), (ii), (iii) we get, $2(a+b+c) = 24-3d$

$\Rightarrow 2(a+b+c+d) = 24-d$

$\Rightarrow a+b+c+d = 12-\frac{d}{2} \quad \ldots\ldots (vi)$

From (i), $d = 9-b-c$
" (ii), $d = 8-c-a$
$\therefore 9-b-c = 8-c-a$
$\Rightarrow a-b = -1 \quad \ldots\ldots (vii)$

Adding (v), (vi), (vii), we get, $2(a+c) = 10-\frac{d}{2}$

$\Rightarrow a+c = 5-\frac{d}{4} \quad \ldots\ldots (viii)$

Now, from (ii) & (viii), $8-d = 5-\frac{d}{4}$

$\Rightarrow d-\frac{d}{4} = 3 \Rightarrow \frac{3d}{4} = 3 \Rightarrow d = 4$

From (v), $c = d-1 = 3$

From (viii), $a = 5-\frac{d}{4}-c = 5-1-3 = 1$

From (vii), $b = a+1 = 1+1 = 2$

$\therefore a = 1, b = 2, c = 3, d = 4$ (Ans).

If $A = \begin{pmatrix} 1 & 0 \\ -1 & 7 \end{pmatrix}$ then find the value of k such that $A^2 = 8A + kI$

$$A = \begin{pmatrix} 1 & 0 \\ -1 & 7 \end{pmatrix}$$

$$\therefore A^2 = \begin{pmatrix} 1 & 0 \\ -1 & 7 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ -1 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -8 & 49 \end{pmatrix}$$

Given That,

$$A^2 = 8A + K \cdot I$$

$$\text{or, } \begin{pmatrix} 1 & 0 \\ -8 & 49 \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ -8 & 56 \end{pmatrix} + \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}$$

$$\text{or, } \begin{pmatrix} 1 & 0 \\ -8 & 49 \end{pmatrix} = \begin{pmatrix} 8+K & 0 \\ -8 & 56+K \end{pmatrix}$$

$$\therefore 8 + K = 1$$

$$\Rightarrow K = -7.$$

If $A = \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}$ (where ω is imaginary cube root of 1), find A^{100}

If $A = \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}$, where '$\omega$' is the imaginary cube root of 1. Find A^{100}.

$$A = \omega \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \omega I$$

$$\therefore A^{100} = (\omega I)^{100} = \omega^{100} \cdot I$$

$$= \omega I \qquad \left[\because \omega^{100} = \omega^{99} \cdot \omega = (\omega^3)^{33} \cdot \omega = \omega\right]$$

$$= A$$

If $A = \begin{pmatrix} 0 & 7 \\ 0 & 0 \end{pmatrix}$ and $f(x) = 1 + x + x^2 + \cdots + x^{20}$, then find $f(A)$.

$$f(x) = 1 + x + x^2 + \cdots + x^{20}$$

$$\therefore f(A) = I + A + A^2 + \cdots + A^{20}$$

Now, $A = \begin{pmatrix} 0 & 7 \\ 0 & 0 \end{pmatrix}$

$$\therefore A^2 = \begin{pmatrix} 0 & 7 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 7 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\therefore A^3 = A^4 = \cdots = A^{20} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

$$\therefore f(A) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 7 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} 1 & 7 \\ 0 & 1 \end{pmatrix}$$

If $2A - 3B = \begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix}$ and $3A + 2B = \begin{pmatrix} -1 & 2 \\ 0 & 4 \end{pmatrix}$, then find the matrices A and B.

If $2A - 3B = \begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix}$ and $3A+2B = \begin{pmatrix} -1 & 2 \\ 0 & 4 \end{pmatrix}$

find the matrices A & B.

From 1st condition, (multiply with 3)

$$6A - 9B = \begin{pmatrix} 3 & 12 \\ 6 & 24 \end{pmatrix} \quad \ldots\ldots (i)$$

From 2nd condition, (multiply with 2)

$$6A + 4B = \begin{pmatrix} -2 & 4 \\ 0 & 8 \end{pmatrix} \quad \ldots\ldots (ii)$$

(ii) − (i), we get,

$$13B = \begin{pmatrix} -5 & -8 \\ -6 & -16 \end{pmatrix} \text{ or, } B = -\frac{1}{13}\begin{pmatrix} 5 & 8 \\ 6 & 16 \end{pmatrix}$$

Similarly, find matrix A. $A = \frac{1}{13}\begin{pmatrix} -1 & 14 \\ 4 & 28 \end{pmatrix}$

If $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ then show that $A^2 - 4A + 3I = O$. Hence find A^{-1}.

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\therefore A^2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix}$$

$$4A = 4\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 8 & -4 \\ -4 & 8 \end{pmatrix}$$

$$3I = 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

$$\therefore A^2 - 4A + 3I$$

$$= \begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix} - \begin{pmatrix} 8 & -4 \\ -4 & 8 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

(Proved)

$$\therefore A^2 - 4A + 3I = O$$

Multiplying by A^{-1} (both sides),

$$A^{-1}.A^2 - 4(A^{-1}.A) + 3(A^{-1}.I) = A^{-1}.O$$

$$\boxed{\begin{aligned} &A^{-1}.A^2 \\ &= (A^{-1}.A)A \\ &= I.A \\ &= A \end{aligned}}$$

or, $A - 4I + 3A^{-1} = O$

or, $3A^{-1} = 4I - A$

or, $3A^{-1} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} - \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

or, $A^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

If $A = \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}$, then show that $I + A = (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$$A = \begin{pmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{pmatrix}$$

L.H.S.,

$$I + A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{pmatrix} = \begin{pmatrix} 1 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 1 \end{pmatrix}$$

R.H.S,

$$(I - A)\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \tan\frac{\alpha}{2} \\ -\tan\frac{\alpha}{2} & 1 \end{pmatrix}\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

$$= \begin{pmatrix} \cos\alpha + \tan\frac{\alpha}{2}\cdot\sin\alpha & -\sin\alpha + \tan\frac{\alpha}{2}\cos\alpha \\ -\tan\frac{\alpha}{2}\cdot\cos\alpha + \sin\alpha & \tan\frac{\alpha}{2}\cdot\sin\alpha + \cos\alpha \end{pmatrix}$$

Now, $\cos\alpha + \tan\frac{\alpha}{2}\cdot\sin\alpha$

$$= \cos\alpha + \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}\cdot 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} = 1 - 2\sin^2\frac{\alpha}{2} + 2\sin^2\frac{\alpha}{2} = 1$$

$$-\sin\alpha + \tan\frac{\alpha}{2}\cdot\cos\alpha = \sin\frac{\alpha}{2}\left(-2\cos\frac{\alpha}{2} + \frac{2\cos^2\frac{\alpha}{2} - 1}{\cos\frac{\alpha}{2}}\right) = -\tan\frac{\alpha}{2}$$

$$\sin\alpha - \tan\frac{\alpha}{2}\cdot\cos\alpha = \sin\frac{\alpha}{2}\left(2\cos\frac{\alpha}{2} - \frac{2\cos^2\frac{\alpha}{2} - 1}{\cos\frac{\alpha}{2}}\right) = \tan\frac{\alpha}{2}$$

$$= \begin{pmatrix} 1 & -\tan\alpha/2 \\ \tan\frac{\alpha}{2} & 1 \end{pmatrix}$$

$\therefore$ L.H.S. = R.H.S. (Proved)

If $A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ then show that $A^2 - 2A + I_2 = O$; Hence find A^{50}

$$A^2 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

$\therefore A^2 - 2A + I_2$

$$= \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ -2 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O$$

$\therefore A^2 = 2A - I \quad \ldots\ldots \text{(i)}$

$$A^3 = A^2 \cdot A = (2A - I)\cdot A = 2A^2 - A$$
$$= 4A - 2I - A \quad \text{(from i)}$$
$$= 3A - 2I$$

Now, $A^4 = A^3 \cdot A = (3A - 2I)\cdot A = 3A^2 - 2A$

$$= 3(2A - I) - 2A \quad \text{(from i)}$$
$$= 4A - 3I$$

$$\therefore A^{50} = 50A - 49I = \begin{pmatrix} 50 & 0 \\ -50 & 50 \end{pmatrix} - \begin{pmatrix} 49 & 0 \\ 0 & 49 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -50 & 1 \end{pmatrix} \quad \text{[Ans]}$$

If $A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ then show that $A^2 = 3A - 2I$. Hence show that $A^8 = 255A - 254I$

$$A^2 = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix}$$

$$3A - 2I = \begin{pmatrix} 3 & 0 \\ 3 & 6 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix}$$

$$\therefore A^2 = 3A - 2I$$

$$A^3 = \begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 7 & 8 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 1 & 0 \\ 7 & 8 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 15 & 16 \end{pmatrix}$$

Let,

$$\therefore A^n = \begin{pmatrix} 1 & 0 \\ 2^n - 1 & 2^n \end{pmatrix} \quad \ldots\ldots (i)$$

(i) is true, for $n = 1, 2, 3, 4, \ldots\ldots$

Let, (i) is true for $n = m$.

$$\therefore A^{m+1} = \begin{pmatrix} 1 & 0 \\ 2^m - 1 & 2^m \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2^{m+1} - 1 & 2^{m+1} \end{pmatrix}$$

So, (i) is true, for $n = m+1$.

So, (i) is true for all $n \in \mathbb{N}$ (mathematical induction)

$$\therefore A^8 = \begin{pmatrix} 1 & 0 \\ 255 & 256 \end{pmatrix}$$

$$255A - 254I = \begin{pmatrix} 255 & 0 \\ 255 & 510 \end{pmatrix} - \begin{pmatrix} 254 & 0 \\ 0 & 254 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 255 & 256 \end{pmatrix}$$

Proved

Prove that every square matrix can be expressed as a sum of a symmetric matrix & a skew-symmetric matrix.

Let, A be a square matrix.

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) = P + Q$$

where; $P = \frac{1}{2}(A + A^T)$ and $Q = \frac{1}{2}(A - A^T)$

Now, $P^T = \left\{\frac{1}{2}(A + A^T)\right\}^T = \frac{1}{2}\left\{A^T + (A^T)^T\right\}$

$\Rightarrow P^T = \frac{1}{2}(A^T + A) = \frac{1}{2}(A + A^T)$

$\Rightarrow P^T = P$

$\Rightarrow P$ is a symmetric matrix.

$Q^T = \left\{\frac{1}{2}(A - A^T)\right\}^T = \frac{1}{2}\left\{A^T - (A^T)^T\right\}$

$\Rightarrow Q^T = \frac{1}{2}(A^T - A) = -\frac{1}{2}(A - A^T)$

$\Rightarrow Q$ is a skew symmetric matrix.

(Proved)

Express a matrix $\begin{pmatrix} -3 & 4 & 1 \\ 2 & 3 & 0 \\ 1 & 4 & 5 \end{pmatrix}$ as the sum of a symmetric matrix and a skew-symmetric matrix.

We know that, for any square matrix A,

$$A = \frac{1}{2}\left(A + A^T\right) + \frac{1}{2}\left(A - A^T\right)$$

Here, $\frac{1}{2}\left(A + A^T\right)$ is a symmetric matrix, and $\frac{1}{2}\left(A - A^T\right)$ is a skew-symmetric "

$$A = \begin{pmatrix} -3 & 4 & 1 \\ 2 & 3 & 0 \\ 1 & 4 & 5 \end{pmatrix} \therefore A^T = \begin{pmatrix} -3 & 2 & 1 \\ 4 & 3 & 4 \\ 1 & 0 & 5 \end{pmatrix}$$

$$\therefore \frac{1}{2}\left(A + A^T\right) = \begin{pmatrix} -3 & 3 & 1 \\ 3 & 3 & 2 \\ 1 & 2 & 5 \end{pmatrix}$$

$$\frac{1}{2}\left(A - A^T\right) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}$$

$$\therefore \begin{pmatrix} -3 & 4 & 1 \\ 2 & 3 & 0 \\ 1 & 4 & 5 \end{pmatrix} = \begin{pmatrix} -3 & 3 & 1 \\ 3 & 3 & 2 \\ 1 & 2 & 5 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}$$

If $P = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$, then show that $P^2 = P$ and then find a matrix Q such that $3P^2 - 2P + Q = I$

$$P = \begin{pmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{pmatrix} \times \begin{pmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1+3-5 & -3-9+15 & -5-15+25 \\ -1-3+5 & 3+9-15 & +5+15-25 \\ 1+3-5 & -3-9+15 & -5-15+25 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{pmatrix} = P.$$

Given, $3P^2 - 2P + Q = I$

$$\Rightarrow 3P - 2P + Q = I$$
$$(\because P^2 = P)$$

$$\Rightarrow Q = I - P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{pmatrix}$$

$$\Rightarrow Q = \begin{pmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{pmatrix}$$

Given $A = \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{pmatrix}$ and $f(x) = x^2 - 5x + 6$; Find $f(A)$

$\therefore f(A) = A^2 - 5A + 6I$

Now, $A^2 = \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{pmatrix} \times \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 4+1 & -1 & 2 \\ 4+2+3 & 1-3 & 2+3 \\ 2-2 & -1 & 1-3 \end{pmatrix}$

$= \begin{pmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{pmatrix}$

$\therefore f(A) = \begin{pmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{pmatrix} - \begin{pmatrix} 10 & 0 & 5 \\ 10 & 5 & 15 \\ 5 & -5 & 0 \end{pmatrix} + \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}$

$= \begin{pmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{pmatrix}$

Show that $A = \begin{pmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{pmatrix}$ is a nilpotent matrix.

$$A^2 = \begin{pmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{pmatrix} \times \begin{pmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} 1+3-4 & -3-9+12 & -4-12+16 \\ -1-3+4 & 3+9-12 & 4+12-16 \\ 1+3-4 & -3-9+12 & -4-12+16 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So, A is a nilpotent matrix of index 2.

If $A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$, then show that $B^T(AB)$ is a diagonal matrix.

$$B = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \therefore B^T = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1-1 & -1+1 \\ -1 & -1+2 & 1-2+1 \\ 0 & 1 & -1+1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Now, $B^T.(AB)$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\because$ All off-diagonal entries of $B^T.(AB)$ are zero, then $B^T.(AB)$ is a diagonal matrix.

If $(x \quad 4 \quad 1)\begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & 2 \\ 0 & 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ 4 \\ -1 \end{pmatrix} = O$, then find the value of x.

$$\begin{pmatrix} x & 4 & 1 \end{pmatrix}_{1\times 3}\begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & 2 \\ 0 & 2 & -4 \end{pmatrix}_{3\times 3}$$

$$= \begin{pmatrix} 2x+4 & x+2 & 2x+4 \end{pmatrix}$$

given expression (L.H.S.)

Now, $= \begin{pmatrix} 2x+4 & x+2 & 2x+4 \end{pmatrix}_{1\times 3} \times \begin{pmatrix} x \\ 4 \\ -1 \end{pmatrix}_{3\times 1}$

$$= \begin{pmatrix} 2x^2+4x+4x+8-2x-4 \end{pmatrix}_{1\times 1}$$

$$= \begin{pmatrix} 2x^2+6x+4 \end{pmatrix}_{1\times 1} = O = \begin{pmatrix} 0 \end{pmatrix}_{1\times 1}$$ [as per given condition]

$\therefore\ 2x^2+6x+4 = 0$

$\Rightarrow 2x^2+4x+2x+4 = 0$

$\Rightarrow 2x(x+2)+2(x+2) = 0$

$\Rightarrow 2(x+2)(x+1) = 0$

$\therefore\ x = -1, -2.$

Show that the matrix $P = \frac{1}{3}\begin{pmatrix} -1 & 2 & -2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$ is an orthogonal matrix. Hence find P^{-1}

$\therefore P.P^T$

$$= \frac{1}{9}\begin{pmatrix} -1 & 2 & -2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} \times \begin{pmatrix} -1 & -2 & 2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 1+4+4 & 2+2-4 & -2+4-2 \\ 2+2-4 & 4+1+4 & -4+2+2 \\ -2+4-2 & -4+2+2 & 4+4+1 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix} = I$$

Similarly, we can show that, $P^T.P = I$.

$\therefore$ P is an orthogonal matrix.

So, $P^{-1} = P^T = \frac{1}{3}\begin{pmatrix} -1 & -2 & 2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{pmatrix}$

If $A=\begin{pmatrix}6 & 2 & -2\\ -2 & 2 & 2\\ 2 & 2 & 2\end{pmatrix}$ then show that $(A-2I)(A-4I)=O$; Hence find A^3

$$A-2I=\begin{pmatrix}6 & 2 & -2\\ -2 & 2 & 2\\ 2 & 2 & 2\end{pmatrix}-\begin{pmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{pmatrix}$$

$$=\begin{pmatrix}4 & 2 & -2\\ -2 & 0 & 2\\ 2 & 2 & 0\end{pmatrix}$$

$$A-4I=\begin{pmatrix}6 & 2 & -2\\ -2 & 2 & 2\\ 2 & 2 & 2\end{pmatrix}-\begin{pmatrix}4 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 4\end{pmatrix}=\begin{pmatrix}2 & 2 & -2\\ -2 & -2 & 2\\ 2 & 2 & -2\end{pmatrix}$$

$\therefore (A-2I)(A-4I)$

$$=\begin{pmatrix}4 & 2 & -2\\ -2 & 0 & 2\\ 2 & 2 & 0\end{pmatrix}\times\begin{pmatrix}2 & 2 & -2\\ -2 & -2 & 2\\ 2 & 2 & -2\end{pmatrix}$$

$$=\begin{pmatrix}8-4-4 & 8-4-4 & -8+4+4\\ -4+4 & -4+4 & 4-4\\ 4-4 & 4-4 & -4+4\end{pmatrix}=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

Now, $(A-2I)(A-4I)=O$

$$\Rightarrow A^2-6A+8I=O \Rightarrow A^2=6A-8I$$

$$\therefore A^3=(6A-8I).A=6A^2-8A=6(6A-8I)-8A$$

$$=36A-48I-8A$$

$$=28A-48I$$

$$\therefore A^3=\begin{pmatrix}120 & 56 & -56\\ -56 & 8 & 56\\ 56 & 56 & 8\end{pmatrix}$$ (Do it yourself)

If $A \cdot adj(A) = \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}$ then find $|A|$

We know that, $A \cdot adj(A) = |A| \cdot I$

$A \cdot adj(A) = \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix} = 10 \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\therefore |A| = 10.$

If A is a symmetric matrix, then A^{-1} is – (a) symmetric, (b) skew-symmetric, (c) diagonal, (d) scalar matrix.

$(A^{-1})^T = (A^T)^{-1} = A^{-1}$ [$\because A$ is symmetric]

$\therefore A^{-1}$ is a symmetric matrix.

If A is a symmetric matrix, then A^n is ________________ matrix.

$\because (A^n)^T = (A^T)^n = A^n$. $\therefore A^n$ is symmetric matrix.

If A is a skew-symmetric matrix, then A^n is – (a) symmetric, (b) skew-symmetric, (c) none of these matrices.

$\because A$ is a skew-symmetric matrix, $\therefore A^T = -A$.

Now, $(A^n)^T = (A^T)^n = (-A)^n = \begin{cases} A^n, & \text{if } n \text{ even} \\ -A^n, & \text{if } n \text{ odd.} \end{cases}$

Answer: (c) None of these.

If A and B are two matrices such that $AB = B$ and $BA = A$. Then $A^2 + B^2$ equals to – (a) $2AB$, (b) AB, (c) $2BA$, (d) $A + B$

$A^2 + B^2 = A \cdot A + B \cdot B = A \cdot (BA) + B \cdot (AB)$
$= (AB) \cdot A + (BA) \cdot B = BA + AB$
$= A + B$

If A is a square matrix, then $A - A^T$ is – (a) symmetric, (b) skew-symmetric, (c) scalar matrix.

$\therefore (A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T)$

so, $(A - A^T)$ is a skew-symmetric matrix.

If a matrix A is both symmetric and skew-symmetric, then A is – (a) diagonal, (b) identity, (c) zero, (d) none of these matrices.

As, A is a symmetric matrix, $\therefore A^T = A$,
" A is a skew-symmetric ", $\therefore A^T = -A$,

$\therefore A = -A$
$\Rightarrow 2A = O \quad \Rightarrow A = O$

$\therefore$ A is a zero matrix,

If A, B are two symmetric matrices of same order, then show that – (i) $AB + BA$ is a symmetric matrix, (ii) $AB - BA$ is a skew-symmetric matrix.

$(AB+BA)^T = (AB)^T + (BA)^T$

$= B^T A^T + A^T B^T$

$= BA + AB$ [∵ A, B are symmetric matrices]

$= AB + BA$ [∵ Matrix addition is commutative]

∴ $(AB+BA)$ is a symmetric matrix.

$(AB-BA)^T = (AB)^T - (BA)^T$

$= B^T A^T - A^T B^T$

$= BA - AB$ [∵ A, B are symmetric matrices]

$= -(AB - BA)$

∴ $(AB - BA)$ is a skew-symmetric matrix.

Show that $B^T AB$ is symmetric/skew-symmetric matrix, if A is symmetric/skew-symmetric matrix.

$$(B^T A B)^T = B^T A^T (B^T)^T = B^T A^T B.$$

So, $(B^T A B)^T = B^T A^T B = \begin{cases} B^T A B, & \text{if } A \text{ is symmetric,} \\ -B^T A B, & \text{if } A \text{ is skew-} \text{''} \end{cases}$