ORDINARY DIFFERENTIAL EQUATION

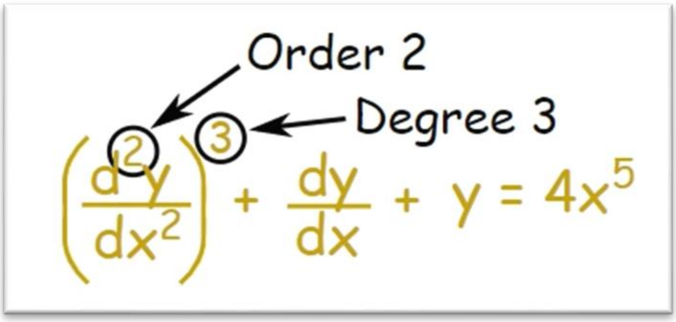
An ordinary differential equation is an equation that contains one independent variable (x) and/or one dependent variable (y) and its one/more derivates (e.g. $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ny}{dx^n}, \dots$) . E.g.

i. $y + \frac{dy}{dx} = 5x$

ii. $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = xe^x$

iii. $\frac{x\,dy - y\,dx}{x^2+y^2} + x\,dx + y\,dy = 0$

Order & Degree of a differential equation:



- The order of a differential equation is the order of the highest order derivative present in the equation.
- The degree of a differential equation is the highest power of highest order derivative when the differential equation is expressed in polynomial equation in derivatives, i.e. index of all derivatives should be positive integer. Derivatives do not come within trigonometric ratios (sin, cos, tan, …), log or can't be in the power of any number. As in this case, we can't convert given differential equation to a polynomial equation in derivatives.
- Every differential equation must have an order, but degree may or may not be defined, because we can't always covert given differential equation in polynomial form.

Differential Equation	Differential equation converted to polynomial equation	Order	Degree
$\frac{dy}{dx} = e^x$		1	1
$\frac{d^2y}{dx^2} = 6y$		2	1
$\frac{d^3y}{dx^3} + x^2\left(\frac{d^2y}{dx^2}\right)^3 = 0$		3	1
$\frac{dy}{dx} + 1 = \sqrt{1+y}$		1	1
$\left[1 + \left(\frac{dy}{dx}\right)^3\right]^{\frac{7}{3}} = 7\left(\frac{d^2y}{dx^2}\right)$	$\left[1 + \left(\frac{dy}{dx}\right)^3\right]^7 = 343\left(\frac{d^2y}{dx^2}\right)^3$	2	3
$y = x\frac{dy}{dx} + a\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$	$(x^2 - a^2)\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} + (y^2 - a^2) = 0$	1	2
$\frac{d^3y}{dx^3} + y^2 + e^{\frac{dy}{dx}} = 0$	$\frac{dy}{dx}$ is in $e^{\frac{dy}{dx}}$, so we can't convert it to polynomial equation.	3	Undefined
$\left(\frac{d^2y}{dx^2}\right)^{\frac{3}{2}} - \left(\frac{dy}{dx}\right)^{\frac{1}{2}} - 4 = 0$	$\left\{\left(\frac{d^2y}{dx^2}\right)^3 - \frac{dy}{dx} + 16\right\}^2 = 64\left(\frac{d^2y}{dx^2}\right)^3$	2	6
$\frac{d^4y}{dx^4} - \sin\left(\frac{d^3y}{dx^3}\right) = 0$	$\frac{d^3y}{dx^3}$ is in $\sin\left(\frac{d^3y}{dx^3}\right)$, so we can't convert it to polynomial equation.	4	Undefined
$\frac{d^2y}{dx^2} + \log\left(\frac{dy}{dx}\right) = y$	$\frac{dy}{dx}$ is in $\log(\frac{dy}{dx})$, so we can't convert it to polynomial equation.	2	Undefined

$$\left(\frac{d^2y}{dx^2}\right)^{\frac{3}{2}} - \left(\frac{dy}{dx}\right)^{\frac{1}{2}} - 4 = 0$$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)^{\frac{3}{2}} - 4 = \left(\frac{dy}{dx}\right)^{\frac{1}{2}}$$

$$\Rightarrow \left[\left(\frac{d^2y}{dx^2}\right)^{\frac{3}{2}} - 4\right]^2 = \frac{dy}{dx}$$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)^3 - 8\left(\frac{d^2y}{dx^2}\right)^{\frac{3}{2}} + 16 = \frac{dy}{dx}$$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)^3 - \frac{dy}{dx} + 16 = 8\left(\frac{d^2y}{dx^2}\right)^{\frac{3}{2}}$$

$$\Rightarrow \left[\left(\frac{d^2y}{dx^2}\right)^3 - \frac{dy}{dx} + 16\right]^2 = \left\{8\left(\frac{d^2y}{dx^2}\right)^{\frac{3}{2}}\right\}^2$$

$$\Rightarrow \left[\left(\frac{d^2y}{dx^2}\right)^3 - \frac{dy}{dx} + 16\right]^2 = 64\left(\frac{d^2y}{dx^2}\right)^3$$

BASIC DISCUSSION ON DIFFERENTIAL EQUATION:

Let's start with an example.

Take an equation $y = A\sin x + B\cos x$ where A, B are arbitrary constants. We've to form differential equation from this.

We already know that differential equation contains independent variable and/or dependent variable & its derivatives, but does not contains any arbitrary constants. So, we start by differentiating the given equation with respect to x & keep in mind that we've to remove all arbitrary constants.

In this respect, we have to remember following three points, while we form differential equation from any given equation –

1. We've to check that – "is it possible to reduce the number of arbitrary constants?" If possible, we've to reduce it.
2. Order of the differential equation should be same as number of arbitrary constants.
3. We should remove arbitrary constant, but does not remove fixed constant. We discuss about constants later on our discussion.

Now we back to our problem.

$$y = A\sin x + B\cos x \Rightarrow \frac{dy}{dx} = A\cos x - B\sin x$$

$$\Rightarrow \frac{d^2y}{dx^2} = -A\sin x - B\cos x = -(A\sin x + B\cos x) = -y \Rightarrow \frac{d^2y}{dx^2} + y = 0$$

So, the differential equation of $y = A\sin x + B\cos x$ is $\frac{d^2y}{dx^2} + y = 0$

Conversely, if we solve the differential equation $\frac{d^2y}{dx^2} + y = 0$, we get $y = A\sin x + B\cos x$.

So, we can say that $y = A\sin x + B\cos x$ is a solution of $\frac{d^2y}{dx^2} + y = 0$.

As before, when we solve any differential equation, we should remember that –

- When we solve any differential equation, number of arbitrary constants within the solution should be same as the order of that differential equation.

Now we come to know that, if we solve any differential equation, we get its solution. This solution is of two types –

i. General solution: if solution contains arbitrary constants. E.g. $y = A\sin x + B\cos x$
ii. Particular solution: if we put value to the arbitrary constants, we get particular solution. E.g. if we take $A = 1$ and $B = 2$ in $y = A\sin x + B\cos x$, we get $y = \sin x + 2\cos x$. Then $y = \sin x + 2\cos x$ is called particular solution of $\frac{d^2y}{dx^2} + y = 0$

FORMATION OF DIFFERENTIAL EQUATION OF KNOWN GEOMETRIC PROBLEMS –

1. Find differential equation of all circles whose centre at origin.

The equation of a circle with centre at origin is $x^2 + y^2 = r^2$ where r is radius. So, by changing the value of r, we get equation of all circles with centre at origin. So, the value of r, which is constant for one circle, varies circle by circle. This r is arbitrary constant. If we get some quantity, which is fixed for all circles, we call this as fixed constant.

So, we start differentiating $x^2 + y^2 = r^2$ with respect to x. We get $2x + 2y\frac{dy}{dx} = 0$

or, $x + y\frac{dy}{dx} = 0$ is the required differential equation of family of circles with centre at origin.

2. Find the differential equation of all circles of radius r and whose centre lies on the line $y = x$

The equation of all circles, whose radius r and centre lying on the line $y = x$, is

$(x-a)^2 + (y-a)^2 = r^2$, where a is an arbitrary constant & r is fixed constant. Differentiating with respect to x, we get $2(x-a) + 2(y-a)\frac{dy}{dx} = 0$

$\Rightarrow a = \frac{x+yy_1}{1+y_1}$ where $y_1 = \frac{dy}{dx}$

Putting value of a within $(x-a)^2 + (y-a)^2 = r^2$ and simplifying we get

$(x-y)^2(1+y_1^2) = r^2(1+y_1)^2$ which is the required differential equation.

3. Find differential equation of all circles which touch x-axis at origin.

Equation of all circles, which touch x-axis at origin is $(x-0)^2 + (y-c)^2 = c^2$ where c is an arbitrary constant. Simplifying it we get, $x^2 + y^2 = 2yc \cdots$ (i)

Differentiating (i) w.r.t. x, we get $2x + 2y\frac{dy}{dx} = 2c\frac{dy}{dx} = \frac{x^2+y^2}{y}\frac{dy}{dx}$ [from (i)]

Simplifying we get, $(y^2 - x^2)\frac{dy}{dx} + 2xy = 0$ which is the required differential equation.

Form the differential equation by removing arbitrary constants.

$y = A\,e^{2x} + B\,e^{-2x} + x$ (A, B arbitrary constants)	$\frac{d^2y}{dx^2} - 4y + 4x = 0$
$x = A\cos nt + B\sin nt$ (A, B arbitrary & n fixed constants)	$\frac{d^2x}{dt^2} + n^2x = 0$
$(x-a)^2 + (y-b)^2 = r^2$ (a, b arbitrary & r fixed constants)	$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = \pm r\,\frac{d^2y}{dx^2}$
$y^2 = A(B-x)(B+x)$ (A, B arbitrary constants)	$xy\,\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$
$y = (A+Bx)e^{-kx}$ (A, B arbitrary & k fixed constants)	$\frac{d^2y}{dx^2} + 2k\frac{dy}{dx} + k^2y = 0$
$y = Ax + \frac{B}{x}$ (A, B arbitrary constants)	$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$

Now we're going to solve differential equations of 1st order & 1st degree. They are –

1) Method of separation of variables
2) Method of substitution
3) Solving homogeneous function by substituting $y = vx$ or $x = vy$
4) Solving by convert it to perfect differential
5) Solving a linear differential equation using integrating factor.

Method of separation of variables

The general form of differential equation (of 1st order and 1st degree) can be written as

$\frac{dy}{dx} = f(x, y)$ or $M\,dx + N\,dy = 0 \cdots \cdots \cdots (i)$ (where M, N are functions of x & y)

If we convert this differential equation to $f_1(x)\,dx + f_2(y)\,dy = 0 \cdots \cdots \cdots (ii)$, then we see that the variables x, y are separated into two functions namely $f_1(x)$ and $f_2(y)$.

Now we easily solve the differential equation (ii) by integrating its both sides, as following –

$\int f_1(x)\,dx + \int f_2(y)\,dy = c \cdots \cdots \cdots (iii)$ (where c is an arbitrary constant)

No (iii) is the general solution of differential equation (i) and this method is known as method of separation of variables. If we put the value of arbitrary constant in (iii), then we get particular solution of (i).

Method of substitution

Sometimes we can't directly convert given differential equation into variable separation form. We substitute some part of given differential equation by some variable, so that substituted differential equation can be converted into variable separation form.

Solving homogeneous function by substituting $y = vx$ or $x = vy$

- A function, $f(x)$, is said to be a homogeneous function of degree n, if we can express the said function as $f(tx) = t^n f(x)$
- A function, $f(x, y, \dots)$, is said to be a homogeneous function of degree n, if we can express the said function as $f(tx, ty, \dots) = t^n f(x, y, \dots)$

E.g. $f(x, y) = \frac{x^2+y^2}{2xy}$ is a homogeneous function of degree 0

To solve a homogeneous differential equation of 1st order & 1st degree, we may use following substitutions: $y = vx$ (or) $x = vy$ and then substituted differential equation can be solved by variable separation process.

Solve by convert to perfect differential

- $M\,dx + N\,dy$ (where M, N are functions of x, y) is said to be a perfect differential, if we can find a function $u(x, y)$ such that $du = M\,dx + N\,dy$
- A differential equation of type $M(x, y)dx + N(x, y)dy = 0$ is called an exact differential equation, if we can convert $M(x, y)dx + N(x, y)dy$ to perfect differentials. We can easily solve an exact differential equation by integrating both sides.

$d(x \pm y)$	$dx \pm dy$
$d(xy)$	$xdy + ydx$
$d\left(\frac{x}{y}\right)$	$\frac{ydx - xdy}{y^2}$
$d\left(\frac{y}{x}\right)$	$\frac{xdy - ydx}{x^2}$
$d(x^2)$	$2xdx$
$d(y^2)$	$2ydy$
$d\left(\frac{x^2 + y^2}{2}\right)$	$xdx + ydy$
$d\left(\frac{x^2}{y}\right)$	$\frac{yd(x^2) - x^2dy}{y^2} = \frac{2xydx - x^2dy}{y^2}$
$d\left(\frac{y^2}{x}\right)$	$\frac{xd(y^2) - y^2dx}{x^2} = \frac{2xydy - y^2dx}{x^2}$
$d(\sin^{-1} x)$	$\frac{dx}{\sqrt{1 - x^2}}$
$d(\cos^{-1} x)$	$-\frac{dx}{\sqrt{1 - x^2}}$
$d(\tan^{-1} x)$	$\frac{dx}{1 + x^2}$
$d(\sin^{-1} xy)$	$\frac{d(xy)}{\sqrt{1 - x^2y^2}} = \frac{xdy + ydx}{\sqrt{1 - x^2y^2}}$
$d(\cos^{-1} xy)$	$-\frac{d(xy)}{\sqrt{1 - x^2y^2}} = -\frac{xdy + ydx}{\sqrt{1 - x^2y^2}}$

$d(\tan^{-1} xy)$	$\frac{d(xy)}{1+x^2y^2} = \frac{xdy + ydx}{1+x^2y^2}$
$d\left(\sin^{-1}\frac{x}{y}\right)$	$\frac{d\left(\frac{x}{y}\right)}{\sqrt{1-\frac{x^2}{y^2}}} = \frac{\frac{ydx - xdy}{y^2}}{\sqrt{\frac{y^2-x^2}{y^2}}} = \frac{ydx - xdy}{y\sqrt{y^2-x^2}}$
$d\left(\sin^{-1}\frac{y}{x}\right)$	$\frac{d\left(\frac{y}{x}\right)}{\sqrt{1-\frac{y^2}{x^2}}} = \frac{\frac{xdy - ydx}{x^2}}{\sqrt{\frac{x^2-y^2}{x^2}}} = \frac{xdy - ydx}{x\sqrt{x^2-y^2}}$
$d\left(\cos^{-1}\frac{x}{y}\right)$	$-\frac{d\left(\frac{x}{y}\right)}{\sqrt{1-\frac{x^2}{y^2}}} = -\frac{\frac{ydx - xdy}{y^2}}{\sqrt{\frac{y^2-x^2}{y^2}}} = -\frac{ydx - xdy}{y\sqrt{y^2-x^2}}$
$d\left(\cos^{-1}\frac{y}{x}\right)$	$-\frac{d\left(\frac{y}{x}\right)}{\sqrt{1-\frac{y^2}{x^2}}} = -\frac{\frac{xdy - ydx}{x^2}}{\sqrt{\frac{x^2-y^2}{x^2}}} = -\frac{xdy - ydx}{x\sqrt{x^2-y^2}}$
$d\left(\tan^{-1}\frac{x}{y}\right)$	$\frac{d\left(\frac{x}{y}\right)}{1+\frac{x^2}{y^2}} = \frac{\frac{ydx - xdy}{y^2}}{\frac{y^2+x^2}{y^2}} = \frac{ydx - xdy}{x^2+y^2}$
$d\left(\tan^{-1}\frac{y}{x}\right)$	$\frac{d\left(\frac{y}{x}\right)}{1+\frac{y^2}{x^2}} = \frac{\frac{xdy - ydx}{x^2}}{\frac{x^2+y^2}{x^2}} = \frac{xdy - ydx}{x^2+y^2}$
$d(\log x)$	$\frac{dx}{x}$
$d(\log xy)$	$\frac{d(xy)}{xy} = \frac{xdy + ydx}{xy}$
$d\left(\log\frac{x}{y}\right)$	$\frac{d\left(\frac{x}{y}\right)}{\frac{x}{y}} = \frac{\frac{ydx - xdy}{y^2}}{\frac{x}{y}} = \frac{ydx - xdy}{xy}$
$d\left(\log\frac{y}{x}\right)$	$\frac{d\left(\frac{y}{x}\right)}{\frac{y}{x}} = \frac{\frac{xdy - ydx}{x^2}}{\frac{y}{x}} = \frac{xdy - ydx}{xy}$

Solving a linear differential equation using integrating factor

- A differential equation of type $\frac{dy}{dx} + Py = Q$ (where P, Q are either continuous functions of x or constants) is called linear differential equation of 1st order in y.
- A differential equation of type $\frac{dx}{dy} + Px = Q$ (where P, Q are either continuous functions of y or constants) is called linear differential equation of 1st order in x.
- Bernoulli's Equation: A differential equation of type $\frac{dy}{dx} + Py = Qy^n$ (where P, Q are continuous functions of x and n is a real number other than 0 and 1) is called Bernoulli's equation.

Integrating Factor (I.F.) - A integrating factor is a function that we multiply both sides of the non-exact differential equation to convert it an exact differential equation.

$\frac{dy}{dx} + Py = Q$ (where P, Q are either continuous functions of x or constants) $\cdots \cdots (i)$

Here Integrating Factor is $= e^{\int P dx}$. Multiplying both sides of (i) with this I.F. we get,

$$\frac{d}{dx}\left(ye^{\int P dx}\right) = Qe^{\int P dx}$$

$\Rightarrow ye^{\int P dx} = \int\left(Qe^{\int P dx}\right) dx + C$ (where C is a constant of integration)

$\frac{dx}{dy} + Px = Q$ (where P, Q are either continuous functions of y or constants) $\cdots \cdots (ii)$

Here Integrating Factor is $= e^{\int P dy}$. Multiplying both sides of (ii) with this I.F. we get,

$$\frac{d}{dy}\left(xe^{\int P dy}\right) = Qe^{\int P dy}$$

$\Rightarrow xe^{\int P dy} = \int\left(Qe^{\int P dy}\right) dy + C$ (where C is a constant of integration)

Solve non-linear differential equation:

To solve a non-linear differential equation, we've to convert the equation to linear differential equation [of form (i) or (ii)] by proper substitution. We show below step by step.

- First, we've to format the given non-linear differential equation as $f'(y)\frac{dy}{dx} + P \cdot f(y) = Q$ (where P, Q are functions of x) $\cdots \cdots (iii)$
- Then if we substitute $f(y) = z$, equation (iii) converts to $\frac{dz}{dx} + Pz = Q$ which is linear differential equation in z.

Solve Bernoulli's equation:

We can easily solve Bernoulli's equation by above mentioned process of solution of non-linear differential equation. See below …

$\frac{dy}{dx} + Py = Qy^n$ (where P, Q are continuous functions of x and $n \in \mathbb{R}$ such that $n \neq 0,1$)

$\Rightarrow y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$ (dividing both sides by y^n)

[$y^{1-n} = z$ (let), then $(1-n)y^{-n}\frac{dy}{dx} = \frac{dz}{dx}$]

$\Rightarrow \frac{dz}{dx} + (1-n)Pz = (1-n)Q$, which is a linear differential equation in z

$\frac{dy}{dx} = -\frac{x(1+y^2)}{y(1+x^2)}$ (given $y = 1$, when $x = 0$)

We can write given differential equation as

$y(1 + x^2)\, dy + x(1 + y^2)\, dx = 0$

$\Rightarrow \frac{x\,dx}{1+x^2} + \frac{y\,dy}{1+y^2} = 0$. Integrating both sides we get,

$\int \frac{x\,dx}{1+x^2} + \int \frac{y\,dy}{1+y^2} = k$ (where k is constant of integration)

$\Rightarrow \frac{1}{2}\log(1 + x^2) + \frac{1}{2}\log(1 + y^2) = \frac{1}{2}\log c$ (where we assume $k = \frac{1}{2}\log c$ and $c > 0$)

$\Rightarrow \log(1 + x^2)(1 + y^2) = \log c$

$\Rightarrow (1 + x^2)(1 + y^2) = c \;\cdots\cdots\cdots (i)$ (General solution)

Putting $x = 0$ and $y = 1$ in equation no. (i), we get $c = 2$

So, the particular solution of given differential equation is $(1 + x^2)(1 + y^2) = 2$

$\frac{dy}{dx} = \frac{e^x(y^2+1)}{y(e^x+1)}$ (given $y = 0$ when $x = 0$)

We can write the given differential equation as (variable separation form)

$\frac{y\,dy}{y^2+1} - \frac{e^x\,dx}{e^x+1} = 0$. Integrating both sides we get,

$\int \frac{y\,dy}{y^2+1} - \int \frac{e^x\,dx}{e^x+1} = k$ (where k is constant of integration)

$\Rightarrow \frac{1}{2}\log(y^2 + 1) - \log(e^x + 1) = \frac{1}{2}\log c$ (where we assume $k = \frac{1}{2}\log c$ and $c > 0$)

$\Rightarrow y^2 + 1 = c(e^x + 1)^2 \cdots\cdots\cdots (i)$

Putting $x = 0$ and $y = 0$ in (i), required particular solution is $y^2 + 1 = \frac{1}{4}(e^x + 1)^2$

$y - x\frac{dy}{dx} = 2\left(1 + x^2\frac{dy}{dx}\right)$ (given that $y = 1$ when $x = 1$)

When we go to solve any differential equation of 1st order – 1st degree, we try to convert the given differential equation into variable separation form.

From the given differential equation, we get,

$$(y - 2) = (2x^2 + x)\frac{dy}{dx}$$

$$\Rightarrow \int \frac{dx}{x(2x+1)} = \int \frac{dy}{y-2} + k \text{ (where } k \text{ is constant of integration)}$$

Required particular solution is $\left|\frac{x}{2x+1}\right| = \frac{1}{3}|y - 2|$

$\tan^{-1}\left(\frac{dy}{dx}\right) = x + y$

Here we do the following substitution to convert the given differential equation to variable separation form.

Let, $x + y = z \cdots (i)$

Differentiate (i) w.r.t. x, we get $1 + \frac{dy}{dx} = \frac{dz}{dx}$

So, given differential equation is now converted into,

$$\frac{dy}{dx} = \tan(x + y)$$

$$\Rightarrow \frac{dz}{dx} - 1 = \tan z$$

$$\Rightarrow \frac{dz}{dx} = 1 + \tan z = \frac{\cos z + \sin z}{\cos z}$$

$$\therefore \int \frac{\cos z}{\cos z + \sin z} dz = \int dx + k \text{ (where } k \text{ is constant of integration)}$$

After integration, the required general solution is

$$y - x + \log|\sin(x + y) + \cos(x + y)| = c$$