Determinant

What is determinant?

- For every square matrix, there exists a number (may be real or complex). This number is called determinant of that matrix. The determinant of matrix A is denoted by $det(A)$ or $det\ A$ or $|A|$ or Δ.
- Let A be a square matrix of 2. E.g., $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then determinant of this matrix is $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Let $\quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a matrix of order 2×2,

then the determinant of A is defined as:

$$\det(A) = |A| = \Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

- Let A be a square matrix of order 3. E.g., $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. To find its determinant, we express this determinant as sum of 2×2 determinants as follow. This process is called expansion of determinant.

The determinant of a square matrix of order 3

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{aligned}
\Rightarrow \det(A) &= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \text{ (first row expansion)} \\
&= a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} \text{ (second row expansion)} \\
&= a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} \text{ (third row expansion)} \\
&= a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} \text{ (first column expansion)} \\
&= a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32} \text{ (second column expansion)} \\
&= a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33} \text{ (third column expansion)}
\end{aligned}$$

Note: If elements of a row (or column) are multiplied with cofactors of any other row (or column), then their sum is zero. E.g., $a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = 0$

Here A_{ij} is the cofactor of a_{ij}, and we can expand a determinant along any column or any row as we desire. Normally we expand determinant along that row / column which has maximum number of zeroes.

$$\text{Minor of } a_{11} = M_{11} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}\, a_{33} - a_{23}\, a_{32}$$

$$\text{Minor of } a_{21} = M_{21} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = a_{12}\, a_{33} - a_{32}\, a_{13}$$

$$\boxed{\textit{Cofactor of } a_{ij} = A_{ij} = (-1)^{i+j}.\ M_{ij}}$$

$$\text{Cofactor of } a_{11} = A_{11} = (-1)^{1+1}.\ M_{11} = (-1)^2.\ (a_{22}\, a_{33} - a_{32}\, a_{23})$$
$$= a_{22}\, a_{33} - a_{32}\, a_{23}$$

$$\text{Cofactor of } a_{21} = A_{21} = (-1)^{2+1}.\ M_{21} = (-1)^3.\ (a_{12}\, a_{33} - a_{32}\, a_{13})$$
$$= (-1).\ (a_{12}\, a_{33} - a_{32}\, a_{13}) = a_{32}\, a_{13} - a_{12}\, a_{33}$$

- Notes: Sign pattern for cofactors

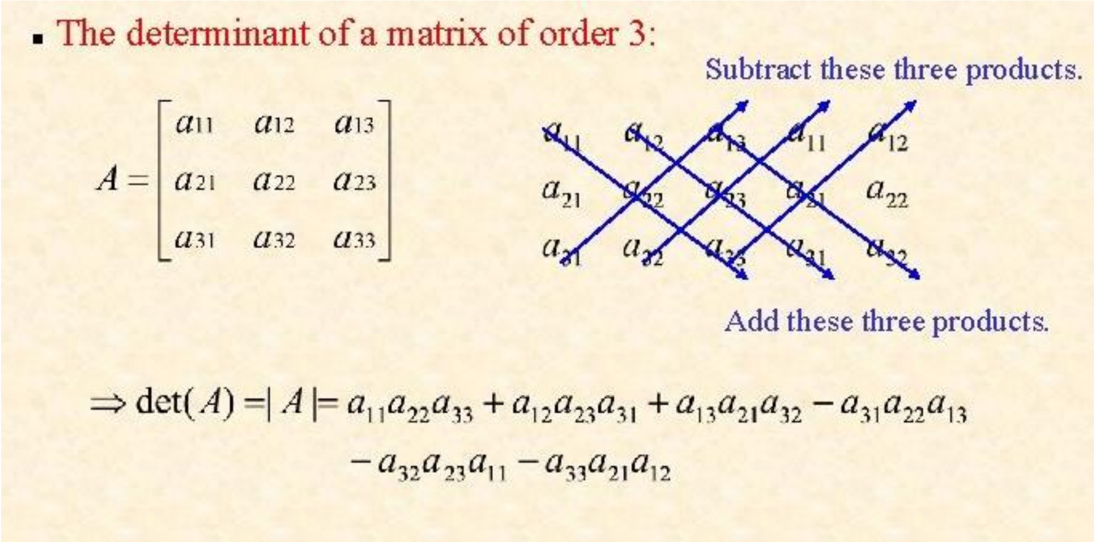
$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \qquad \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

3×3 matrix $\qquad$ 4×4 matrix

Example: If $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$, then show that $|2A| = 4|A|$

$2A = \begin{pmatrix} 2 & 4 \\ 8 & 4 \end{pmatrix}$; $\therefore\ |2A| = \begin{vmatrix} 2 & 4 \\ 8 & 4 \end{vmatrix} = 8 - 32 = -24$ and $|A| = \begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} = 2 - 8 = -6$

- Quickly find out the value of determinant of order 3 without expansion:



<u>Properties of determinant</u>:

- $|kA| = k^n \times |A|$ where k is any scalar & A is a square matrix of order n.
- $|AB| = |A| \times |B|$ where A & B are two square matrices of same order.
- $\because A \times A^{-1} = I$, so

  $|A \times A^{-1}| = |I|$

  $\Rightarrow \; |A| \times |A^{-1}| = 1$

  $\Rightarrow \; |A| = \frac{1}{|A^{-1}|}$
- The value of determinant remains unchanged if its rows & columns are interchanged, i.e., $\det(A) = \det\,(A^T)$

$$\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix} = \begin{vmatrix} 2 & 6 & 1 \\ -3 & 0 & 5 \\ 5 & 4 & -7 \end{vmatrix}$$

- If any two rows (or columns) of a determinant are interchanged, then sign of determinant changes. E.g. if $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta_1 = \begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix}$ (where 1st row and 3rd row are interchanged). Here $\Delta = -\Delta_1$

- If any two rows (or columns) of a determinant are identical (i.e., all corresponding elements are same), then value of determinant is zero.

  E.g., $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} = 0$ as 1st row & 3rd row are identical.

  $$\begin{vmatrix} 3 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 2 & 3 \end{vmatrix} = 0$$

- If all elements of a row (or column) are zero, then determinant is zero.

  $$\begin{vmatrix} 0 & 0 & 0 \\ 3 & 4 & 5 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

- If each element of a row (or a column) of a determinant is multiplied by a constant k, then its value gets multiplied by k. E.g., if $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and

  $\Delta_1 = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, then $\Delta_1 = k \times \Delta$. Again

  $$\begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix} = \begin{vmatrix} 6(17) & 6(3) & 6(6) \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix} = 6 \times \begin{vmatrix} 17 & 3 & 6 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$$

- If all elements of a row (or a column) of a determinant are expressed as sum of two (or more) terms, then the determinant can be expressed as sum of two (or more) determinants. E.g.

  $$\begin{vmatrix} a_1+\lambda & b_1+\lambda & c_1+\lambda \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \lambda & \lambda & \lambda \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

- If, to each element of any row (or column) of a determinant, the equimultiples of corresponding elements of other row (or column) are added, then value of determinant remains the same, i.e., the value of determinant remains same if we apply the operation $R_i \to R_i + kR_j$ or $C_i \to C_i + kC_j$

E.g., $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + ka_3 & b_1 + kb_3 & c_1 + kb_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ $(R_1 \to R_1 + kR_3)$

Again,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + \alpha b_1 + \beta c_1 & b_1 & c_1 \\ a_2 + \alpha b_2 + \beta c_2 & b_2 & c_2 \\ a_3 + \alpha b_3 + \beta c_3 & b_3 & c_3 \end{vmatrix} \quad (C_1 \to C_1 + \alpha C_2 + \beta C_3)$$

- If in a determinant all the elements above or below the diagonal is zero, then value of the determinant is equal to product of the diagonal elements.

$$\begin{vmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & i \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ d & e & 0 \\ g & h & i \end{vmatrix} = a \times e \times i$$

- Value of determinant of skew-symmetric matrix is 0, if its order is odd. E.g.,

$$\begin{vmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{vmatrix} = 0$$

- If a determinant Δ becomes zero when we put $x = a$, then $(x - a)$ is a factor of determinant Δ.

<u>Area of triangle using determinant</u>:

Let the coordinates of three vertices of a triangle are $(x_1, y_1), (x_2, y_2)$ & (x_3, y_3). Then the area of this triangle is <u>absolute value</u> of $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ sq. units

Example: Find value of k, if area of triangle is 4 sq. unit and vertices are $(k, 0), (4, 0)$& $(0,2)$

Area = $\frac{1}{2}\begin{vmatrix} k & 0 & 1 \\ 4 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix} = \pm 4$

Expand along column 2, we get $\frac{1}{2}\left\{-2 \times \begin{vmatrix} k & 1 \\ 4 & 1 \end{vmatrix}\right\} = \pm 4$

$\Rightarrow \ (k - 4) = \mp 4$

So, values of k are 0 & 8